Laboratory handouts, ME 340

This document contains summary theory, solved exercises, prelab assignments, lab instructions, and report assignments for Lab 5.

Laboratory handout 4 – Second-order systems

In this handout, material and assignments marked as *optional* can be skipped when preparing for the lab, but may provide a useful resource at other times during the course.

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An **exponentially decaying harmonic** signal is of the form

$$x(t) = Ae^{-\gamma t}\cos(\omega t - \theta) \tag{1}$$

in terms of an **initial amplitude** $A > 0$, a **decay rate** $\gamma > 0$, an **angular frequency** $\omega > 0$, and a **phase shift** θ.
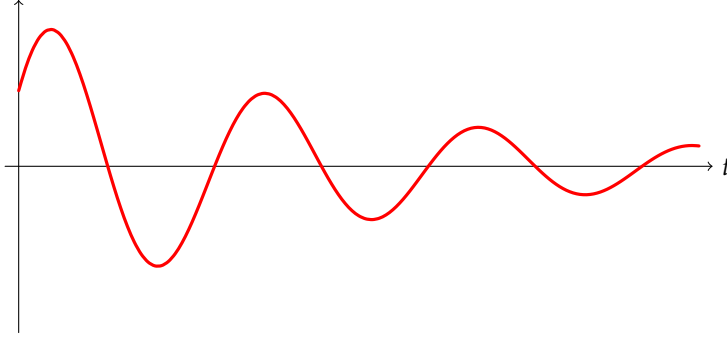

Figure 1: An exponentially decaying harmonic signal.

The signal crosses 0 for some $t = t^*$, provided that

$$\omega t^* = \theta + \frac{\pi}{2} + n\pi, \tag{2}$$

for some integer n. Since

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = -A\sqrt{\omega^2 + \gamma^2}e^{-\gamma t}\cos\left(\omega t - \theta - \arctan\frac{\omega}{\gamma}\right), \tag{3}$$

the signal has a sequence of local maxima and minima for $t = t^{**}$, where

$$\omega t^{**} = \theta + \arctan\frac{\omega}{\gamma} + \frac{\pi}{2} + n\pi, \tag{4}$$

for some integer n.

The elapsed time between consecutive maxima (minima) of $x(t)$ equals $2\pi/\omega$. The ratio between the values of $x(t)$ at consecutive

maxima (minima) equals the **logarithmic decrement** $e^{2\pi\gamma/\omega}$. The absolute value of the ratio between the values of $x(t)$ at consecutive extrema equals $e^{\pi\gamma/\omega}$.

$$\text{———————}\quad\bullet\quad\text{———————}$$

If measurements on a physical system result in a signal $x(t)$ that is dominated by the solution to a differential equation of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t) + 2\zeta\omega_n\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \omega_n^2 x(t) = f(t) \tag{5}$$

for some $\omega_n, \zeta > 0$ and some given function $f(t)$, then the system is said to be **linear**, **second-order**, **time-invariant**, and **stable**. The constants ω_n and ζ are the **natural frequency** and **damping ratio**, respectively.

The **unit impulse response** of the system is given by the convolution

$$h(t) = \left(e^{\gamma_1 \#} * e^{\gamma_2 \#}\right)(t) := \int_0^t e^{\gamma_1 \tau}e^{\gamma_2(t-\tau)}\,\mathrm{d}\tau \tag{6}$$

where $u = \gamma_1$ and $u = \gamma_2$ are the roots of the **characteristic polynomial**

$$u^2 + 2\zeta\omega_n u + \omega_n^2 = 0, \tag{7}$$

i.e.,

$$\gamma_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \mathrm{j}\omega_n\sqrt{1 - \zeta^2}. \tag{8}$$

The roots lie in the complex plane and are either purely real or complex with imaginary parts of opposite sign.

In particular, for $0 < \zeta < 1$, the system is **underdamped** and

$$h(t) = \frac{e^{\gamma_1 t} - e^{\gamma_2 t}}{\gamma_1 - \gamma_2} = \frac{e^{-\zeta\omega_n t}}{\omega_d}\sin\omega_d t, \tag{9}$$

where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ is the **damped natural frequency**. In this case, the unit impulse response is an exponentially decaying harmonic signal with initial amplitude $1/\omega_d$, decay rate $\zeta\omega_n$ and angular frequency ω_d.
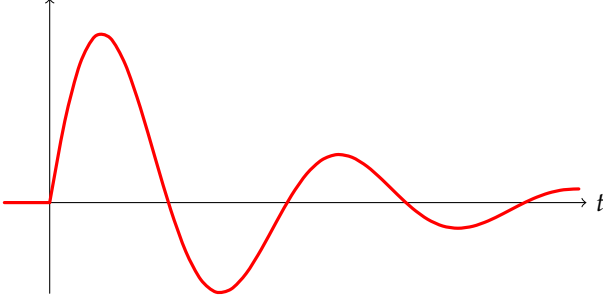
Figure 2: The unit impulse response of an underdamped second-order system.

For $\zeta > 1$, the system is **overdamped** and

$$h(t) = \frac{e^{\gamma_1 t} - e^{\gamma_2 t}}{\gamma_1 - \gamma_2} = \frac{e^{-\zeta \omega_n t}}{\omega_n \sqrt{\zeta^2 - 1}} \sinh\left(\omega_n \sqrt{\zeta^2 - 1}\, t\right). \qquad (10)$$

In this case, the unit impulse response is a sum of exponentially decaying terms, and does not oscillate about 0.
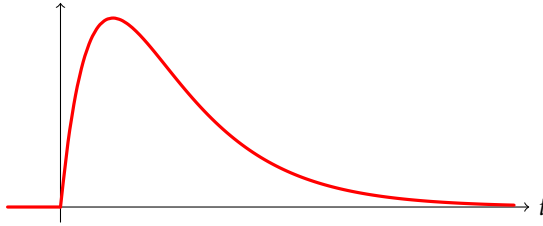

Figure 3: The unit impulse response of an overdamped second-order system.

Finally, for $\zeta = 1$, the system is **critically damped** and

$$h(t) = t e^{-\omega_n t}. \qquad (11)$$

In all three cases, $h(0) = 0$ and $\mathrm{d}h/\mathrm{d}t(0) = 1$.

$\qquad\qquad\qquad \bullet \qquad\qquad\qquad$

For arbitrary $f(t)$, the corresponding solution to (5) is given by

$$x(t) = \left(2\zeta\omega_n x(0) + \frac{\mathrm{d}x}{\mathrm{d}t}(0)\right) h(t) + x(0)\frac{\mathrm{d}h}{\mathrm{d}t}(t) + \left(f(\#) * h(\#)\right)(t), \qquad (12)$$

where

$$\left(f(\#) * h(\#)\right)(t) := \int_0^t f(\tau) h(t - \tau)\, \mathrm{d}\tau. \qquad (13)$$

If $f(t) = 0$ for $t \geq 0$, the solution

$$x(t) = \left(2\zeta\omega_n x(0) + \frac{\mathrm{d}x}{\mathrm{d}t}(0) \right) h(t) + x(0)\frac{\mathrm{d}h}{\mathrm{d}t}(t) \qquad (14)$$

is a **free response** of the system. In particular, for $0 < \zeta < 1$ and $\mathrm{d}x/\mathrm{d}t(0) = 0$,

$$x(t) = x(0)\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\left(\omega_d t - \frac{\pi}{2} + \arccos\zeta \right). \qquad (15)$$

If $f(t) = c$ for $t \geq 0$ and $x(0) = \mathrm{d}x/\mathrm{d}t(0) = 0$, the solution

$$x(t) = c\int_0^t h(\tau)\,\mathrm{d}\tau \qquad (16)$$

is a **step response** of the system. In particular, for $0 < \zeta < 1$,

$$x(t) = \frac{c}{\omega_n^2}\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\left(\omega_d t - \frac{\pi}{2} + \arccos\zeta \right) \right). \qquad (17)$$

For $t \gg 1$, this is approximately equal to the **steady-state response** $x_{ss}(t) = c/\omega_n^2$. The difference $x_{ss} - x(t)$ is an exponentially decaying harmonic signal that equals the free response with $x(0) = c/\omega_n^2$ and $\mathrm{d}x/\mathrm{d}t(0) = 0$.
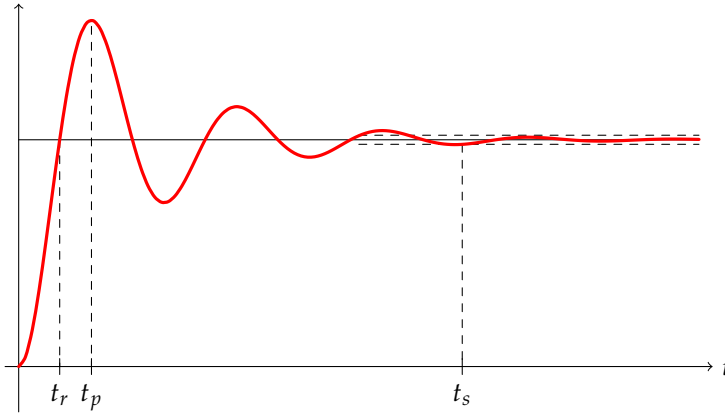

Figure 4: A step response for an under-damped system, including the rise time t_r, the peak time t_p, and the settling time t_s.

If the natural frequency ω_n and damping ratio ζ of a linear, second-order, time-invariant, stable system are unknown, they may

be estimated from measurements. Such an approach is an example of **system** or **plant identification**.

If it is possible to engineer the system so that $f(t) = 0$ for $t \geq 0$, then ω_n and ζ may be estimated from the free response with $dx/dt(0) = 0$. The system is underdamped if this is oscillatory as in (15), and overdamped otherwise[1].

In the underdamped case, the damping ratio ζ may be estimated from the absolute value of the ratio of two consecutive extrema, since this equals $e^{\pi\zeta/\sqrt{1-\zeta^2}}$, or from the logarithmic decrement, since this equals $e^{2\pi\zeta/\sqrt{1-\zeta^2}}$. The natural frequency may be estimated from the time elapsed between consecutive zero crossings or between consecutive extrema, since these both equal π/ω_d.

If it is possible to engineer the system so that $f(t)$ equals the constant c for $t \geq 0$ and $x(0) = dx/dt(0) = 0$, then the natural frequency ω_n may be estimated from the steady-state value $x_{ss} := \lim_{t\to\infty} x(t) = c/\omega_n^2$. The damping ratio ζ may then be estimated from the logarithmic decrement for the difference $x(t) - x_{ss}$, since this equals $e^{2\pi\zeta/\sqrt{1-\zeta^2}}$, or from the elapsed time between consecutive maxima, since this equals $2\pi/\omega_d$. If only a single maximum is clearly visible in an experimental measurement, the damping ratio may be estimated from

1. the **rise time** t_r corresponding to the time that $x(t)$ first crosses x_{ss}, since this equals
$$\frac{\pi - \arccos\zeta}{\omega_d}; \qquad (18)$$

2. the **peak time** t_p corresponding to the time that $x(t)$ first peaks, since this equals π/ω_d;

3. the **maximum relative overshoot** corresponding to the ratio
$$\frac{x(t) - x_{ss}}{x_{ss}} \qquad (19)$$
at the peak time, since this equals $e^{-\pi\zeta/\sqrt{1-\zeta^2}}$; or

4. the **settling time** t_s corresponding to the time after which $x(t)$ remains within 2% of x_{ss}, since this is approximately $4/\zeta\omega_n$.

[1] It is, in practice, impossible to achieve a critically damped system, but for $\zeta \approx 1$, it may be difficult to distinguish between an underdamped and a overdamped system in the presence of measurement noise.

A graphical representation of this quantities is shown in Fig. 4.

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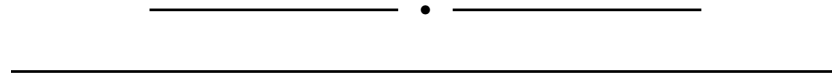
Exercises

1. Express the signal $x(t) = -e^{-t}\cos 2t$ as an exponentially decaying harmonic signal.

2. Find all local extrema of the exponentially decaying harmonic signal $x(t) = e^{-2t}\cos(t - \pi/4)$ and determine the elapsed time between consecutive maxima.

3. Determine the logarithmic decrement for the exponentially decaying harmonic signal $x(t) = 3e^{-t/2}\cos(3t - 2)$ and relate this to the ratio between successive extrema.

4. Measurements on a mechanical system result in a signal that is dominated by the solution to the differential equation

$$20\frac{d^2x}{dt^2}(t) + 10\frac{dx}{dt}(t) + 18x(t) = 2.$$

   Determine whether the system is overdamped or underdamped.

5. For what value of c is the mass-spring-damper system, governed by the differential equation

$$m\frac{d^2x}{dt^2}(t) + c\frac{dx}{dt}(t) + kx(t) = f(t),$$

   critically damped?

6. Find the unit step response of a linear, second-order, time-invariant system with natural frequency 2 and damping ratio 1/2.

7. Compute the rise time, peak time, and settling time for the unit step response of the linear, second-order, time-invariant system with natural frequency $\omega_n = 4$ and damping ratio $\zeta = 0.8$.

8. Suppose that the maximum relative overshoot of the step response of a linear, second-order, time-invariant system equals approximately 0.2. Estimate the damping ratio.

9. Describe the dependence of the rise time on the damping ratio ζ on the interval $[0, 1]$ and determine its minimum value.

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Solutions

1. From the general form of an exponentially decaying harmonic signal, $\gamma = -1$, $\omega = 2$, $A \cos \theta = -1$, and $A \sin \theta = 0$. This implies that $A = 1$ and $\theta = \pi$, i.e., $x(t) = e^{-t} \cos(2t - \pi)$.

2. Here,
$$\frac{dx}{dt}(t) = -\sqrt{5}e^{-2t} \cos\left(t - \frac{\pi}{4} - \arctan\frac{1}{2}\right)$$
and
$$\frac{d^2x}{dt^2}(t) = 5e^{-2t} \cos\left(t - \frac{\pi}{4} - 2\arctan\frac{1}{2}\right).$$

Minima are therefore found at
$$t = \frac{3\pi}{4} + \arctan\frac{1}{2} + 2n\pi$$

and maxima are found at
$$t = -\frac{\pi}{4} + \arctan\frac{1}{2} + 2n\pi.$$

The elapsed time between consecutive maxima equals 2π.

3. Here, the logarithmic decrement equals $e^{\pi/3}$. Extrema occur at
$$t_n = \frac{2}{3} + \frac{\pi}{6} + \frac{1}{3}\arctan 6 + \frac{n\pi}{3}$$

and are separated in time by $\pi/3$. It follows that
$$\frac{x(t_{n+1})}{x(t_n)} = \frac{3e^{-t_n/2}e^{-\pi/6}\cos(3t_n + \pi - 2)}{3e^{-t_n/2}\cos(3t_n - 2)} = -e^{-\pi/6}$$

i.e., the negative square root of the logarithmic decrement.

4. Here,
$$\frac{d^2x}{dt^2}(t) + \frac{1}{2}\frac{dx}{dt}(t) + \frac{9}{10}x(t) = \frac{1}{10},$$

and it follows that $\omega_n^2 = 9/10$ and $2\zeta\omega_n = 1/2 \Rightarrow \zeta = 3/4\sqrt{10} < 1$, i.e., the system is underdamped.

5. Here,
$$2\zeta\omega_n = \frac{c}{m}, \quad \omega_n^2 = \frac{k}{m} \Rightarrow \zeta = \frac{c}{2\sqrt{mk}}$$

The system is critically damped when $c = 2\sqrt{mk}$.

6. Here,
$$h(t) = \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t.$$

If $f(t) = 1$ for $t \geq 0$, the unit step response equals the convolution

$$\left(f(\#) * h(\#)\right)(t) = \frac{1}{4}\left(1 - \frac{2}{\sqrt{3}}e^{-t}\cos\left(\sqrt{3}t - \arctan\frac{1}{\sqrt{3}}\right)\right).$$

7. Here, $\omega_d = 12/5$. It follows that the rise time equals approximately 1.04, that the peak time equals approximately 1.31, and that the settling time equals 1.25.

8. Here,
$$e^{-\pi\zeta/\sqrt{1-\zeta^2}} \approx \mu \Rightarrow \zeta \approx -\frac{\ln\mu}{\sqrt{\pi^2 + \ln^2\mu}}$$

For $\mu = 0.2$, $\zeta \approx 0.46$.

9. The function
$$\zeta \mapsto \frac{\pi - \arccos\zeta}{\omega_n\sqrt{1-\zeta^2}}$$

equals $\pi/2\omega_n$ at $\zeta = 0$ and grows monotonically and without bound as ζ increases to 1. The minimum rise time is $\pi/2\omega_n$.

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Prelab Assignments

Complete these assignments before the lab. Show all work for credit.

1. Express the signal $x(t) = 3e^{-t} \sin 2t$ as an exponentially decaying harmonic signal.

2. Find all local extrema of the exponentially decaying harmonic signal $x(t) = 3e^{-t/2} \cos(3t - 1)$ and determine the elapsed time between consecutive minima.

3. Determine the logarithmic decrement for the exponentially decaying harmonic signal $x(t) = e^{-3t} \cos(t + \pi/2)$ and relate this to the ratio between successive minima.

4. Measurements on a mechanical system result in a signal that is dominated by the solution to the differential equation

$$2\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) + 24\frac{\mathrm{d}x}{\mathrm{d}t}(t) + 18x(t) = 2.$$

Determine whether the system is overdamped or underdamped.

5. For what value of k is the mass-spring-damper system, governed by the differential equation

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) + c\frac{\mathrm{d}x}{\mathrm{d}t}(t) + kx(t) = f(t),$$
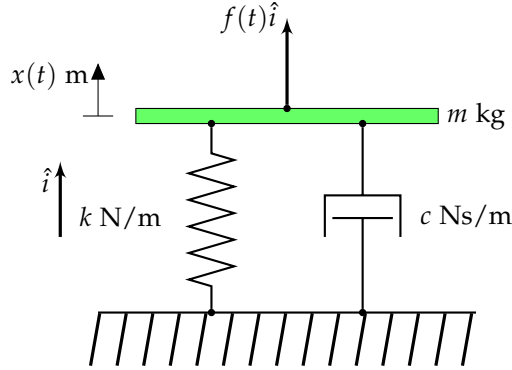
critically damped?

6. Graph the unit step response of a linear, second-order, time-invariant system with natural frequency $\omega_n = 1/2$ and damping ratio $\zeta = 1/\sqrt{2}$.

Lab instructions[2]

Consider the motion of the mechanical suspension shown below, able to translate along the direction described by the vector $\hat{\imath}$.

The displacement of the mass is governed by the differential equation

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}(t) + c\frac{\mathrm{d}x}{\mathrm{d}t}(t) + kx(t) = f(t). \tag{20}$$

This corresponds to a linear, second-order, time-invariant, stable system with natural frequency $\omega_n = \sqrt{k/m}$ (in units of rad/s) and damping ratio $\zeta = c/2\sqrt{km}$. Estimates of the natural frequency and damping ratio based on experimental measurements may be used to estimate the ratios k/m and c/m and, if the mass is known, the stiffness k and damping coefficient c.

In this lab, experiments will be conducted on an Educational Control Products (ECP) Model 210 Rectilinear Dynamic System. This electromechanical apparatus is a three-degree of freedom spring-mass-damper system. It consists of configurable masses on low-friction bearings, springs, dampers and an input drive. Encoders connected to each of the carts determine their locations. The input drive command is generated by a digital to analog (DAC) signal produced by a data acquisition board installed in the benches' PC. The matlab toolbox Real-Time Windows Target will be used to both drive the system when needed and collect response data.

Encoders attached to each cart are used to measure their displacements. These encoders have a resolution of 16,000 counts per rev-

olution. The small pulley that attaches the cart to the encoder has a radius of 1.163 cm, so each cm of displacement is equivalent to $1/(2\pi * 1.163)$ revolutions and, consequently, $16,000/(2\pi * 1.163)$ encoder counts.

To make best use of lab time, look ahead at the Report Assignment and create a table for recording experimental data. Make sure to note your observations for each plot as they are generated.

Free response

In the first experiment, you will use the free response of a linear mass-spring-damper model to explore the dependence of the natural frequency and damping ratios on the values of the mass, stiffness, and damping coefficient.

Follow the detailed instructions below to collect and analyze the free response under each of three experimental conditions:

Condition 1: Small inertia, stiff spring, light damping. On the ECP rectilinear plant at your bench, make sure the second carriage initially has no weights on it. Open the damping adjustment knob 3/4 turn from the fully closed position.

Condition 2: Big inertia, stiff spring, light damping. Add four thick (500 g) brass weights to experimental condition 1.

Condition 3: Big inertia, stiff spring, strong damping. Open the damping adjustment knob 1/8 turn from the fully closed position.

What qualitative changes do you observe in the response when you change the inertia and/or damping of the system?

Experimental procedure

1. Turn on the equipment:

   (a) Start MATLAB using the icon found on your desktop.

   (b) Turn on the ECP control box on the top shelf.

   (c) Inside MATLAB, open the "read-only" file oneDOFinitial.mdl located in the directory N:\labs\me340\Mass_Spring_oneDOF.

(d) Save the file with the name `lab4free<yourNetID>.mdl` in the directory `C:\matlab\me340`.

(e) Change MATLAB's current directory to the location where your model file was saved by typing `cd c:\matlab\me340` at the MATLAB command prompt.

(f) Click the SIMULINK model file to give it focus. Confirm that 0 V is being sent to the DAC output and that data is collected from optical encoder channel 1. Confirm that the gain block is converting encoder counts to distance in cm and that data is sampled by Real-Time Windows Target every 5 ms.

(g) Enter <ctrl+B> on your keyboard to build the model. While your auto-generated code is building return to MATLAB's command prompt. The build is complete when the message "Successful completion of build procedure for model" is displayed.

2. Acquire data:

(a) Simply press the green Run button to start data collection. Note, it can take a number of seconds to start the acquisition so don't click the run button multiple times if the data collection does not start right away.

(b) With your fingers, move cart #2 approximately 1 cm to the right, compressing the spring. Wait for Real-Time Windows Target to refresh the plot window and then release.

(c) Before the plot window is again refreshed (after 5 seconds), click the Stop button to stop data collection.

3. Analyze data:

(a) In MATLAB plot your displacement data:

```
>> data = oneDOFinitial_data;
>> plot(data(:,1), data(:,2));
```

(b) Zoom in on your plot to locate exact data points for estimating ω_n and ζ (for example, successive zeros or extrema), and record

the corresponding values of t and x on a print out of the plot
for future reference.

Step response

In the second experiment, you will use the step response of a linear
mass-spring-damper model to estimate system parameters.

Experimental procedure

1. Turn on the equipment:

    (a) Close the SIMULINK model that you used in the previous
         experiment.

    (b) Open the "read-only" file `oneDOFstep.mdl` located in the direc-
         tory `N:\labs\me340\Mass_Spring_oneDOF`.

    (c) Save the file with the name `lab4step<yourNetID>.mdl` in the
         directory `C:\matlab\me340`

    (d) Change MATLAB's current directory to the location where
         you model file was saved by typing `cd c:\matlab\me340` at the
         MATLAB command prompt.

    (e) Click the SIMULINK model to give it focus. Confirm that the
         "Pulse Generator" amplitude is set to 5. Confirm that the model
         includes a feedback loop that adds damping to the system with
         damping coefficient 5.2 in units of Ns/m.

    (f) Enter <ctrl+B> on our keyboard to build the model. While
         your auto-generated code is building, return to MATLAB's com-
         mand prompt. The build is complete when the message "Suc-
         cessful completion of Real-Time Workshop ..." is displayed.

2. Acquire data:

    (a) Click the Run button to start data collection.

    (b) After several pulses click the Stop button to stop data collec-
         tion.

3. Analyze data:

(a) In MATLAB, plot your displacement data and the input:

```
>> data = oneDOFstep_data;
>> plot(data(:,1), data(:,2), 'r', ...
     data(:,1), data(:,3),'b');
```

(b) Zoom in on your plot to locate data points for estimating ω_n and ζ (see Fig. 4) and record the corresponding values of t and x on a print out of the plot for future reference. Include data points for determining the maximum relative overshoot, the settling time, the peak time, and the rise time.

Impulse response

In the third experiment, you will use the impulse response of a linear mass-spring-damper model to estimate system parameters. In particular, use the value for the stiffness k found in the previous experiment to estimate m and c.

Experimental procedure

1. Initialize experiment:

   (a) Continue to use the SIMULINK model from the previous experiment.

   (b) Change the input from a step input to an approximate impulse by opening the "Pulse Generator" component and setting the amplitude to 20, the period to 10, and the pulse width to 0.5%. This produces a short pulse of 20 N for 50 ms.

2. Acquire data:

   (a) Click the Run button to start data collection.

   (b) After several impulses click the Stop button to stop data collection.

3. Analyze data:

   (a) In MATLAB, plot your displacement data and the input:

```
>> data = oneDOFstep_data;
>> plot(data(:,1), data(:,2), 'r', ...
      data(:,1), data(:,3),'b');
```

(b) Zoom in on your plot to locate data points for estimating ω_n
and ζ and record the corresponding values of t and x on a print
out of the plot for future reference.

Report Assignments

Complete these assignments during the lab. *Show all work for credit.*

1. In the case of the free response, label each plot with the corresponding case and order the plots accordingly.

2. Copy the table below into your report and enter the appropriate data. Show your work used to estimate ω_n and ζ on the plot for the corresponding case, in the case that the system is underdamped. If the system is overdamped, identify this on the plot and in the table.

	Experiment Data Reduction Table						
	Approx. System Parameters			Theoretical Values		Experimental Values	
	c	m	k	ζ	ω_n	ζ	ω_n
Case #	Ns/m	kg	N/m	–	rad/s	–	rad/s
1	8	0.55	780				
2	8	2.55	780				
3	75	2.55	780				

3. In the case of the step response, label each plot accordingly.

4. Determine the maximum relative overshoot, the rise time, the settling time, and the peak time. Use these measurements to estimate ω_n and ζ. Identify the equations used in your calculation on the corresponding plot, and make sure to check the units.

5. Estimate the stiffness k from the steady-state displacement given the pulse amplitude 5 N. Proceed to estimate m and c from the step response. Show your work on the corresponding plot.

6. In the case of the impulse response, label the plot accordingly.

7. Use the value of k obtained from the steady-state response to a step input, and estimate m and c from the impulse response. Show your work on the corresponding plot.