Laboratory handouts, ME 340

This document contains summary theory, solved exercises, prelab assignments, lab instructions, and report assignments for Lab 5.

Laboratory handout 3 – First-order systems

A **signal** is a function of time, denoted by t. The notation $x(t)$ identifies the signal x as a time-dependent function.

An **exponentially decaying** signal is of the form $x(t) = Ce^{-\gamma t}$ in terms of an **initial value** C and a **decay rate** $\gamma > 0$. The signal equals a fraction $1/e$ of its initial value after the characteristic **time scale** $t = 1/\gamma$.

When the **slowest decay rate** $\gamma = \min_{1 \leq i \leq n} \gamma_i$ in a sum of exponentially decaying signals

$$x(t) = C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t} + \cdots + C_n e^{-\gamma_n t} \tag{1}$$

is an order of magnitude smaller than the other decay rates, then the corresponding time scales are **well separated**. In this case, if the initial values $C_1, C_2, \ldots$ are all of the same order of magnitude, then measurements of $x(t)$ for $t > 1/\gamma$ are **dominated** by the component with the slowest decay rate, as long as its magnitude remains above the noise floor.

If measurements on a physical system result in a signal $x(t)$ that is dominated by the solution to a differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \frac{1}{T} x(t) = f(t) \tag{2}$$

for some $T > 0$ and some given function $f(t)$, then the system is said to be **linear**, **first-order**, **time-invariant**, and **stable**.

If $f(t) = 0$ for $t \geq 0$, the solution

$$x(t) = x(0) e^{-t/T} \tag{3}$$

is a **free response** of the system. This is an exponentially decaying signal whose characteristic time scale T is known as the **time constant** of the system.

If $f(t) = 1/T$ for $t \geq 0$ and $x(0) = 0$, the solution

$$x(t) = 1 - e^{-t/T} \tag{4}$$

is a **step response** of the system. For $t \gg 1$, this is approximately equal to the **steady-state response** $x_{ss}(t) = 1$. The difference $x_{ss} - x(t)$ is an exponentially decaying signal with characteristic time scale T.

If the time constant T of a linear, first-order, time-invariant, stable system is unknown, it may be estimated from measurements. Such an approach is an example of **system** or **plant identification**.

If it is possible to engineer the system so that $f(t) = 0$ for $t \geq 0$, then T may be estimated from the characteristic time after which the signal $x(t)$ equals the fraction $1/e$ of its initial value $x(0)$, since in this case

$$x(t) = x(0)e^{-t/T}. \tag{5}$$

If it is possible to engineer the system so that $f(t)$ equals the constant c for $t \geq 0$ and $x(0) = 0$, then T may be estimated from the characteristic time after which the difference between the signal $x(t)$ and its steady-state value $x_{ss} := \lim_{t\to\infty} x(t)$ equals the fraction $1/e$ of its initial value $x(0) - x_{ss}$, since in this case

$$x(t) = cT(1 - e^{-t/T}) \tag{6}$$

and $x_{ss} = \lim_{t\to\infty} x(t) = cT$.

Exercises

1. Determine the decay rate for the exponentially decaying signal $x(t) = 2e^{-t/2}$.

2. After what time does the value of the exponentially decaying signal $x(t) = 3e^{-t/4}$ equal the fraction $1/e$ of its initial value?

3. Determine the slowest decay rate for the sum of exponentially

decaying signals

$$x(t) = 6e^{-t} + \frac{1}{100}e^{-10t} - e^{-t/2}.$$

4. Suppose that the signal

$$x(t) = 4e^{-t/10} + e^{-8t} + 3e^{-2t}$$

is measured in the presence of noise distributed uniformly in the interval $[-0.5, 0.5]$. Are the time scales well separated? Are measurements on the time interval $[10, 20]$ dominated by the component with the slowest decay rate?

5. Find the free response of a linear, first order system with time constant 3 and initial value $5/2$.

6. Find the unit step response of a linear, first order system with time constant 6.

7. Suppose that the response $x(t)$ of a linear, first order system with $f(t) = 0$ for $t \geq 0$ is approximately equal to the fraction $2/e$ of its initial value $x(0)$ after time 2. Estimate the corresponding time constant.

8. Suppose that the response $x(t)$ of a linear, first order system with $f(t)$ constant for $t \geq 0$ and $x(0) = 0$ differs from its steady-state value by approximately a fraction $1/e$ of the initial difference after time 4. Estimate the corresponding time constant.

Solutions

1. Here, $\gamma = 1/2$.

2. Here, $x(4) = 3e^{-1} = x(0)/e$, i.e., at $t = 4$.

3. The slowest decay rate equals $\min\{1, 10, 1/2\} = 1/2$.

4. The slowest decay rate equals $1/10$. The corresponding time scale is 20 times larger than the next faster decay rate, so the decay rates

are well separated. The slowest component is larger than 0.5 as long as $t < 10 \ln 4/0.5 \approx 20.8$. The other two components add up to less than 10^{-8} for $t > 10$. We conclude that the component with slowest decay rate dominates measurements on the interval $[10, 20]$, even in the presence of noise.

5. The free response equals

$$x(t) = \frac{5}{2}e^{-t/3}.$$

6. The unit step response corresponds to the case that $f(t) = 1$ for $t \geq 0$ and $x(0) = 0$. In this case,

$$x(t) = \int_0^t e^{-(t-\tau)/6}\, \mathrm{d}\tau = 6(1 - e^{-t/6}).$$

7. The free response equals

$$x(t) = x(0)e^{-t/T}.$$

It follows that

$$\frac{x(2)}{x(0)} = e^{-2/T} \approx \frac{2}{e} \Rightarrow T \approx \frac{2}{1 - \ln 2} \approx 6.52.$$

8. The step response equals

$$x(t) = cT(1 - e^{-t/T}).$$

and $x_{ss} = \lim_{t\to\infty} x(t) = cT$. It follows that

$$\frac{x(4) - x_{ss}}{x(0) - x_{ss}} = e^{-4/T} \approx \frac{1}{e} \Rightarrow T \approx 4.$$

Prelab Assignments

Complete these assignments before the lab. Show all work for credit.

1. Determine the decay rate for the exponentially decaying signal $x(t) = 4e^{-t/3}$.

2. After what time does the value of the exponentially decaying signal $x(t) = 2e^{-t/6}$ equal the fraction $2/e$ of its initial value?

3. Determine the slowest decay rate for the sum of exponentially decaying signals

$$x(t) = \frac{1}{2}e^{-t/3} - \frac{1}{3}e^{-2t} + 18e^{-t/2}.$$

4. Suppose that the signal

$$x(t) = 2e^{-t/3} + e^{-t} + 3e^{-t/2}$$

is measured in the presence of noise distributed uniformly in the interval $[-0.25, 0.25]$. Are the decay rates well separated? Are measurements dominated by the component with the slowest decay rate on any time interval?

5. Find the free response of a linear, first order system with time constant $3/2$ and initial value 4.

6. Find the two-units step response of a linear, first order system with time constant 3.

7. Suppose that the response $x(t)$ of a linear, first order system with $f(t) = 0$ for $t \geq 0$ is approximately equal to the fraction $1/e$ of its initial value $x(0)$ after time $5/3$. Estimate the corresponding time constant.

8. Suppose that the response $x(t)$ of a linear, first order system with $f(t)$ constant for $t \geq 0$ and $x(0) = 0$ differs from its steady-state value by approximately a fraction $2/e$ of the initial difference after time $3/2$. Estimate the corresponding time constant.

9. A linear, first order system whose response $x(t)$ satisfies the differential equation
$$T\frac{\mathrm{d}x}{\mathrm{d}t}(t) + x(t) = Kf(t)$$
has time constant T and gain K. How would you estimate T and K from measurements of the response $x(t)$ when $f(t) = 1$ for $t \geq 0$?

Lab instructions[1]

[1] These notes are an edited version of handouts authored by Andrew Alleyne.

In this lab, you will investigate two approximately linear, first-order systems. Linear first-order systems describe many physical phenomena. For example, the time history of a capacitor's voltage in a simple RC electrical circuit is described by a linear first- order system. The temperature history of an oven that is losing heat from conduction or convection is also described by a linear first-order system.

A major goal of engineering is to develop mathematical models of physical systems so that these models can be used to predict the **system response** under various **loading conditions**. In control system design, these so-called "plant" models are used as the basis for the formulation of controllers. For example, once an oven's temperature as a function of heat input has been mathematically modeled, one can design a controller to keep the oven at a desired temperature.

An important step in developing such mathematical models is a process called "plant identification" or "identification analysis". In any given mathematical model, there are physical parameters, such as thermal conductivity, time constants, or elastic moduli that must be determined. The plant identification step attempts to generate an experimental estimate for these parameters.

In this lab exercise, numerical values for the time constants of two linear first-order systems will be determined. These time constants will be obtained using two simple methods. In addition, non-ideal behavior will be studied in terms of the actual physical system that is being investigated.

Leaking tank

In the first experiment, you will use the time history of the water level in a leaking tank to estimate the flow resistance across a small outlet. To develop a model that allows us to predict the time history of the water level in the tank, you rely on the following physical principles and assumptions:

- Conservation of mass: the rate of change of mass in the tank equals rate at which mass enters the tank minus the rate at which mass leaves the tank.

- Constant temperature: the density ρ of the water is unchanged.
- Constant cross section: the volume of water is proportional to the water level h.
- The flow is laminar: the mass flow rate through an outlet is proportional to the pressure difference across the outlet.
- The principle of hydrostatic pressure: the pressure at the bottom of the tank exceeds the pressure above the water by $\rho g h$.

These imply that

$$A \frac{\mathrm{d}h}{\mathrm{d}t}(t) + \frac{g}{R} h(t) = q_{in}(t), \tag{7}$$

where A is the cross-sectional area, g is the acceleration of gravity, R is the resistance at the outlet, and $q_{in}(t)$ is the time history of the volume flow rate into the tank. This is a linear, first-order system with time constant $T = AR/g$. If you measure A and estimate T from measurements of the response $x(t)$ to some input $q_{in}(t)$, then you can estimate the resistance R.

Experimental procedure

1. Turn on the equipment:
   (a) Have the TA show you how to turn on the Process Interface.
   (b) Log into the PC and start MATLAB to collect the tank's height data:
      - Start MATLAB using the icon found on your desktop.
      - Inside MATLAB, open the "read-only" file `TankLeak.mdl` located in the directory `N:\HydraulicsLab\TankExperiment`.
      - Save the file with the name `lab1tank<yourNetID>.mdl` in the directory `C:\matlab\me340\`.
      - Change MATLAB's current directory to the location where your model file was saved by typing `cd c:\matlab\me340` at the MATLAB command prompt.

2. Measure the cross-sectional area A of the tank and record this value for later use.

3. Calibrate the water level sensor:

   (a) Consider the height of the water when it is level with the top of the hole in the rubber stopper as $h_{\text{bottom}} = 0$. Measure the vertical distance between h_{bottom} and the 100 mark. Record this value as h_{top} ($\neq 100$).

   (b) Collect calibration data for the float sensor potentiometer.

   - Turn on the Process Interface unit.
   - Make sure the water level in the tank is at the top of the hole in the rubber stopper.
   - Record the value, in mA, that the Digital Display Module reads.
   - The sensor current is passed over a 100 Ω resistor within the Process Interface 38-200, so the above current signal gets converted to a voltage signal, which is sampled by the PC. Convert current to voltage and record it as V_{bottom}.

   (c) Use a finger to block the hole in the rubber stopper between the two tanks.

   (d) Open the manual valve MV2.

   (e) Start the pump using the switch on the extreme lower left of the Process Interface panel.

   (f) When the water level reaches the 100 mark:

   - Close MV2 and turn off the pump.
   - If you exceed the 100 mark, open the drain valve slightly until you are within 1 or 2 mm of the 100 mark.

   (g) Verify that the Digital Display Module reads approximately 20 mA. Calculate the voltage corresponding to reading shown on the Digital Display Module and record this voltage as V_{top}.

   (h) The float sensor is linear, i.e., the relationship between the height of the float sensor and the voltage read by the computer is of the form $h(t) = aV(t) + b$, where a and b are calibration parameters. Since V_{top}, V_{bottom}, h_{top}, and h_{bottom} are all known, find the numerical parameters a and b.

4. Acquire data:

   (a) When you are ready to collect data, press the SIMULINK run button (green arrow). This will start the build process of your SIMULINK model and run the data collection. It will take about 20 seconds for the model to start running.

   (b) When you see data being logged to the SIMULINK Scope window, remove your finger from the exit nozzle to start water flow.

   (c) Observe in the plot window that the computer is recording the height data in terms of voltage vs. time.

   (d) Let the tank drain for the specified 180 seconds. The water level should be just above the hole in the rubber stopper at that point.

   (e) After completion of the 180 second data collection, the sampled data is stored in the MATLAB variable `Height_inVolts`.

5. Analyze data:

   (a) In MATLAB, plot your raw voltage data:

```
>>plot_time = Height_inVolts(:,1);
>>Volts = Height_inVolts(:,2);
>>plot(plot_time, Volts)
```

   (b) Convert the voltage data into height data using the relationship $h(t) = aV(t) + b$ (using your values of a and b):

```
>> height = a*Volts + b;
```

   (c) Plot height vs. time. (The plot should decay to zero if a and b were calculated correctly):

```
>>plot(plot_time, height)
```

   (d) From the plotted data, estimate the time constant T. Assign the corresponding value # to a MATLAB variable:

```
>>tau = #;
```

(e) Compute the theoretical response of the first order system (7):

```
>>thheight = height(1) * exp(-plot_time/tau);
```

(f) Plot the theoretical and experimental height data versus time. Title the plot, label both axes (with units), add a legend, and print it out (use the `doc` command to access the MATLAB help browser for information on how to add a title, labels, and legends):

```
>>plot(plot_time, height, ':', plot_time, thheight, '-')
```

Hydraulic motor

In the second experiment, you will use the time history of the speed $x(t)$ of a hydraulic motor to estimate the gain and time constant of the first-order system

$$T\frac{\mathrm{d}x}{\mathrm{d}t}(t) + x(t) = Ku(t) \tag{8}$$

where $u(t)$ is the command input to a proportional directional control valve. This has the ability to control the direction of the hydraulic fluid flow and the proportion of the flow that gets transmitted through the valve. We assume that the characteristic time scales of the valve and the speed sensor are well separated from the (slower) time scale of the motor. However, such assumptions may not always hold and you must be careful when examining the data.

Experimental procedure

1. Turn on the equipment:

   (a) Confirm that a circuit has been constructed to run the hydraulic motor using the Parker Hannifin BD90 servo valve.

   - The control signal comes from one of the analog output channels on the MW2000 and should already be connected to the co-ax plug labeled EHV Input (command input) on the Parker Hannifin Hydraulic Trainer stand.

- The hydraulic motor's rotational speed is measured via a magnetic proximity sensor, whose output is available for measurement using the co-ax connection located on the side of the big gray box mounted on the trainer stand. It is labeled "tach" and should already be connected to channel 7 on the MW2000, where the frequency signal output by the sensor is converted to a DC voltage that is proportional to the motor speed. In particular, at 2500 rpm, the recorded voltage is 5 V.

(b) If you have not already, close your SIMULINK model from the first experiment.

(c) In MATLAB, open the "read-only" file `motorstep.mdl` located in the directory `N:\HydraulicsLabMotorExperiment`. An empty plot corresponding to the plot of motor speed $x(t)$ and open loop input $u(t)$ should appear. Save the file with the name `lab1motor<yourNetID>.mdl` in the directory `C:\matlab\me340\`.

(d) Confirm that what you have on the screen is an open loop controller that has been developed for you. The purpose of the SIMULINK open loop controller is to generate step inputs to the motor system and to record the output tachometer speed. Notice that the step input is in Volts, and the tachometer output is in rpm. The SIMULINK block diagram allows you to change the magnitude of the step input.

(e) Change MATLAB's current directory to the location you just saved your model file by typing `cd c:\matlab\me340` at the MATLAB command prompt.

(f) Wait for your TA to walk you through the steps required to safely start the hydraulic system.

(g) Make sure that the system pressure is set via the relief valve at 300 psi.

2. Acquire data:

(a) In your SIMULINK block diagram, set the Step block's "Final Value" to 1 V.

(b) Click the SIMULINK start arrow to start the build process. When the build process is finished your step response data will be collected for the given step input. By default the recorded data is saved to a multi-dimensional array called `Velocity_RPM`.

(c) If the recorded data is acceptable (i.e. you see a step response), store the recorded data to MATLAB variables with unique names:

```
>>run25V_time = Velocity_RPM(:,1);
>>run25V_RPM = Velocity_RPM(:,2);
>>run25V_ScaledInput = Velocity_RPM(:,3);
```

(d) Repeat the above data collection for step inputs of 1.50 V and 2 V, but make sure to use unique variable names in the previous step.

3. Analyze data:

(a) Use system identification to estimate the time constant T and steady state gain K for each data set.

(b) For the 1.50 V data set, compute the theoretical first order response given the values for the time constant and gain that you have just obtained.

(c) Compare the theoretical and corresponding experimental response by plotting them on the same axes. You should shift the data so that the response occurs at the same time on the plot. Label the plot appropriately and print it out. Also record the values of T and K on the plot.

Report Assignments

Complete these assignments during the lab. *Show all work for credit.*

1. In the experiment on the leaking tank, you engineered the system to correspond to a particular flow rate $q_{in}(t)$ and initial value $h(0)$, and then used your knowledge of the response $h(t)$ to estimate T. Describe $q_{in}(t)$ and $h(t)$ using the appropriate terminology.

2. Record your measured and estimated values of h_{bottom}, h_{top}, V_{bottom}, V_{top}, a, b, and T with the appropriate units.

3. Use a measurement of the cross-sectional area A and the value $9.81\ \text{m/s}^2$ for the acceleration of gravity to estimate the resistance R across the outlet and record this with the appropriate unit.

4. Plot the theoretical and experimental height data versus time.

5. Critically evaluate the assumptions made in modeling the leaking tank as a linear, first order system over a large range of water levels.

6. In the experiment on the hydraulic motor, you engineered the system to correspond to a particular command input $u(t)$ and initial value $x(0)$, and then used your knowledge of the response $x(t)$ to estimate T. Describe $u(t)$ and $x(t)$ using the appropriate terminology.

7. Record your estimated values of T and K for each of the three step inputs with the appropriate units.

8. Estimate the delay between the step input in the command signal and the system response. What is a possible source of this delay? Does it affect your estimates of T and K?

9. Use the experimental data to critically evaluate the assumption that the hydraulic motor can be modeled as a linear, first order system over a large range of inputs.