Laboratory handout 5 – Mode shapes and resonance

The differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t) + 2\zeta\omega_n\frac{\mathrm{d}x}{\mathrm{d}t}(t) + \omega_n^2x(t) = f(t) \tag{1}$$

describes a **single-degree-of-freedom oscillator** with **damping ratio** $\zeta \geq 0$ and **natural frequency** $\omega_n > 0$.

Suppose that $f(t) = 0$ outside of the interval $[0, \Delta]$. Integration on both sides of (1) yields

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t)\Big|_0^\Delta + 2\zeta\omega_n x(t)\Big|_0^\Delta + \omega_n^2\int_0^\Delta x(\tau)\,\mathrm{d}\tau = \int_0^\Delta f(\tau)\,\mathrm{d}\tau. \tag{2}$$

Provided that $x(t)$ and $\mathrm{d}x(t)/\mathrm{d}t$ are both finite on the interval $[0, \Delta]$, it follows that

$$\lim_{\Delta\to0} x(t)\Big|_0^\Delta = 0,\ \lim_{\Delta\to0}\int_0^\Delta x(\tau)\,\mathrm{d}\tau = 0 \tag{3}$$

and

$$\lim_{\Delta\to0}\frac{\mathrm{d}x}{\mathrm{d}t}(t)\Big|_0^\Delta = \lim_{\Delta\to0}\int_0^\Delta f(\tau)\,\mathrm{d}\tau. \tag{4}$$

If the right-hand side is nonzero, then $f(t)$ is an **impulse**. It follows that an impulse is equivalent to a change in the initial value of $\mathrm{d}x(t)/\mathrm{d}t$. The function $f(t)$ is a **unit impulse** if the limit of the integral equals 1.

For general $f(t)$, the solution to (1) is given by

$$x(t) = \left(2\zeta\omega_n x(0) + \frac{\mathrm{d}x}{\mathrm{d}t}(0)\right)h(t) + x(0)\frac{\mathrm{d}h}{\mathrm{d}t}(t) + \Big(f(\#) * h(\#)\Big)(t), \tag{5}$$

where

$$\Big(f(\#) * h(\#)\Big)(t) := \int_0^t f(\tau)h(t-\tau)\,\mathrm{d}\tau \tag{6}$$

and $h(t)$ is the **unit-impulse response**. Specifically, $h(t)$ is the solution to (1) when $f(t)$ is a unit impulse and $x(0)$ and $\mathrm{d}x/\mathrm{d}t(0)$ both equal 0 or, equivalently, the solution to (1) when $f(t) = 0$, $x(0) = 0$, and $\mathrm{d}x/\mathrm{d}t(0) = 1$.

When $\zeta = 0$, the unit-impulse response equals

$$h(t) = \frac{1}{\omega_n} \sin \omega_n t. \tag{7}$$

In this case, and with $x(0) = 0$, the free response (with $f(t) = 0$) equals

$$x(t) = \frac{1}{\omega_n} \frac{\mathrm{d}x}{\mathrm{d}t}(0) \sin \omega_n t, \tag{8}$$

i.e., a harmonic signal with angular frequency ω_n. Indeed, in this case, the free response is harmonic with angular frequency ω_n for every choice of initial conditions.

When $\zeta > 0$, the system is stable and $h(t)$ is an exponentially decaying signal. For $t \gg 1$, the solution (5) is independent of the initial conditions $x(0)$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(0)$ and only dependent on $f(t)$. In this limit, the solution is called the **steady-state response** to the input $f(t)$ and is denoted by $x_{ss}(t)$.

By substitution in (1), the complex exponential $x_{ss}(t) = Ae^{\mathrm{j}\omega t}$ with (complex) **amplitude** A and **angular frequency** ω is the steady-state response to the input $f(t) = Be^{\mathrm{j}\omega t}$ with (complex) **excitation amplitude** B and **excitation frequency** ω, provided that

$$(-\omega^2 + \mathrm{j}2\zeta\omega_n\omega + \omega_n^2)A = B. \tag{9}$$

For small but nonzero ζ,

$$A \approx \frac{B}{\omega_n^2 - \omega^2} \tag{10}$$

when ω is far from ω_n, and

$$A \approx \frac{B}{\mathrm{j}2\zeta\omega_n^2} \tag{11}$$

when ω is close to ω_n. As seen in the figure on the next page, the dependence of $|A|$ on ω peaks at the **resonance frequency** ω_n. At this excitation frequency, large-in-magnitude steady-state amplitude results even for small-in-magnitude excitation amplitude.

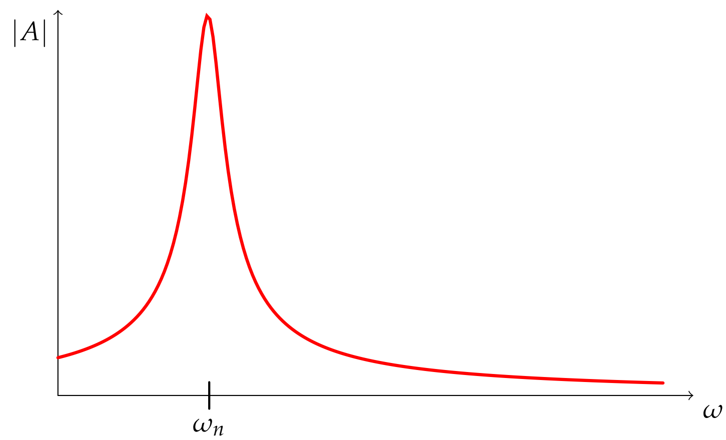


Figure 5: The amplitude of the steady-state response peaks at the resonance frequency equal to the natural frequency ω_n.

For a linear, time-invariant mechanical system with **two degrees of freedom**,

$$M \cdot \begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} + C \cdot \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} + K \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \quad (12)$$

where $\dot{x}_i(t) = \mathrm{d}x_i(t)/\mathrm{d}t$, $\ddot{x}_i(t) = \mathrm{d}^2 x_i(t)/\mathrm{d}t^2$, and

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \quad (13)$$

denote the **mass**, **damping**, and **stiffness matrix**, respectively.

Suppose that $f_1(t)$ and $f_2(t)$ equal 0 outside of the interval $[0, \Delta]$. Integration on the left-hand side of (12) yields

$$M \cdot \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} \Bigg|_0^\Delta + C \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \Bigg|_0^\Delta + K \cdot \int_0^\Delta \begin{pmatrix} x_1(\tau) \\ x_2(\tau) \end{pmatrix} \mathrm{d}\tau \quad (14)$$

Provided that $x_1(t)$, $x_2(t)$, $\dot{x}_1(t)$, and $\dot{x}_2(t)$ are all finite on the interval $[0, \Delta]$, it follows that

$$\lim_{\Delta \to 0} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \Bigg|_0^\Delta = 0, \; \lim_{\Delta \to 0} \int_0^\Delta \begin{pmatrix} x_1(\tau) \\ x_2(\tau) \end{pmatrix} \mathrm{d}\tau = 0 \quad (15)$$

and

$$M \cdot \lim_{\Delta \to 0} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} \Bigg|_0^\Delta = \lim_{\Delta \to 0} \int_0^\Delta \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \mathrm{d}\tau. \tag{16}$$

If the right-hand side is nonzero, then $f_1(t)$ and/or $f_2(t)$ are **impulses**. Each combination of impulses is equivalent to some change in the initial values of $\dot{x}_1(t)$ and $\dot{x}_2(t)$.

Positive values of ω, for which

$$\det\left(K - M\omega^2\right) = 0, \tag{17}$$

are the **natural (modal) frequencies** of the system. For each natural frequency, a column matrix

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \tag{18}$$

such that

$$\left(K - M\omega^2\right) \cdot \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \tag{19}$$

with nonzero A_1 and/or A_2 is a **mode shape** of the system.

In the absence of damping, and with $x_1(0) = x_2(0) = 0$, there are natural frequencies ω_1 and ω_2 and corresponding mode shapes

$$\begin{pmatrix} A_{1,1} \\ A_{2,1} \end{pmatrix} \text{ and } \begin{pmatrix} A_{2,1} \\ A_{2,2} \end{pmatrix} \tag{20}$$

such that the free response equals

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{1}{\omega_1} \begin{pmatrix} A_{1,1} \\ A_{2,1} \end{pmatrix} \sin \omega_1 t + \frac{1}{\omega_2} \begin{pmatrix} A_{1,2} \\ A_{2,2} \end{pmatrix} \sin \omega_2 t, \tag{21}$$

i.e., a sum of harmonic signals with angular frequencies ω_1 and ω_2. Each such free response corresponds to the response (with zero initial conditions) to some combination of impulses $f_1(t)$ and $f_2(t)$.

For arbitrary initial conditions, the free response is again given by (21), but with $\sin \omega_1 t$ and $\sin \omega_2 t$ replaced by $\cos(\omega_1 t - \theta_1)$ and

$\cos(\omega_2 t - \theta_2)$, respectively, for some phase shifts θ_1 and θ_2. If both terms are nonzero, then this is periodic if and only if the ratio ω_1/ω_2 is a rational number.

For $c_{11}, c_{22} > 0$, the mechanical system is stable and the free response is exponentially decaying. For $t \gg 1$, the solution is approximately independent of the initial conditions and only dependent on $f_1(t)$ and $f_2(t)$. In this limit, the solution is called the **steady-state response** and is denoted by $x_{1,ss}(t)$ and $x_{2,ss}(t)$, respectively, for the two degrees of freedom.

By substitution in (12), the complex exponentials $x_{1,ss}(t) = A_1 e^{j\omega t}$ and $x_{2,ss}(t) = A_2 e^{j\omega t}$ with (complex) **amplitudes** A_1 and A_2, and **angular frequency** ω are the steady-state response to the inputs $f_1(t) = B_1 e^{j\omega t}$ and $f_2(t) = B_2 e^{j\omega t}$ with (complex) **excitation amplitudes** B_1 and B_2 and **excitation frequency** ω, provided that

$$(-M\omega^2 + jC\omega + K) \cdot \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \tag{22}$$

In the case of small damping: $\max\{c_{11}, c_{12}, c_{21}, c_{22}\} = \epsilon \ll 1$ and nonzero c_{11} and/or c_{22},

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \approx (K - M\omega^2)^{-1} \cdot \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tag{23}$$

for ω far from any natural frequency, since then $K - M\omega^2$ is an invertible matrix. If, instead, ω is close to a natural frequency ω_i with mode shape

$$\begin{pmatrix} A_{i,1} \\ A_{i,2} \end{pmatrix}, \tag{24}$$

then (outside of exceptional circumstances),

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \approx \frac{k}{\epsilon} \begin{pmatrix} A_{i,1} \\ A_{i,2} \end{pmatrix} \tag{25}$$

for some nonzero k.

As illustrated in the figure on the next page, the dependence of $|A_1|$ and $|A_2|$ on ω peaks at resonance frequencies equal to the natural frequencies. At these excitation frequencies, large-in-magnitude steady-state amplitudes result even for small-in-magnitude excitation amplitudes. At these frequencies, the relative magnitude and phase of the harmonic signals in the two degrees of freedom is described by the corresponding mode shape.

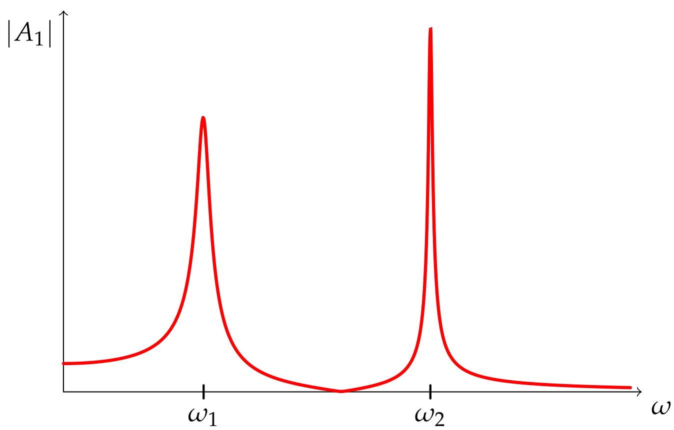


Figure 6: The amplitude of the steady-state response peaks at the resonance frequencies equal to each of the natural frequencies ω_1 and ω_2.

Exercises

1. Determine the natural frequency of the system described by the differential equation

$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t) + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x(t) = 0.$$

2. Write the solution of the initial-value problem

$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t) + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x(t) = 0,\ x(0) = 1,\ \frac{\mathrm{d}x}{\mathrm{d}t}(0) = -1$$

as an exponentially decaying harmonic signal of the form $x(t) = Ae^{-\gamma t}\cos(\omega t - \theta)$.

3. Write the steady-state response of the differential equation

$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t) + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x(t) = e^{\mathrm{j}\omega t}$$

as a complex exponential signal $Ae^{j\omega t}$, and write A in polar form as a function of ω.

4. Show that the natural frequencies and corresponding mode shapes of a linear, time-invariant system with mass matrix M and stiffness matrix K are the square roots of the eigenvalues and the corresponding eigenvectors, respectively, of the matrix $M^{-1} \cdot K$.

5. Find the natural frequencies and mode shapes of a linear, time-invariant mechanical system with

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, K = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

6. Find the solution of the initial-value problem

$$M \cdot \begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} + K \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$x_1(0) = 1,\ x_2(0) = 2,\ \frac{dx_1}{dt}(0) = \frac{dx_2}{dt}(0) = 0$$

and

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, K = \begin{pmatrix} 4 & 8 \\ 8 & 2 \end{pmatrix}.$$

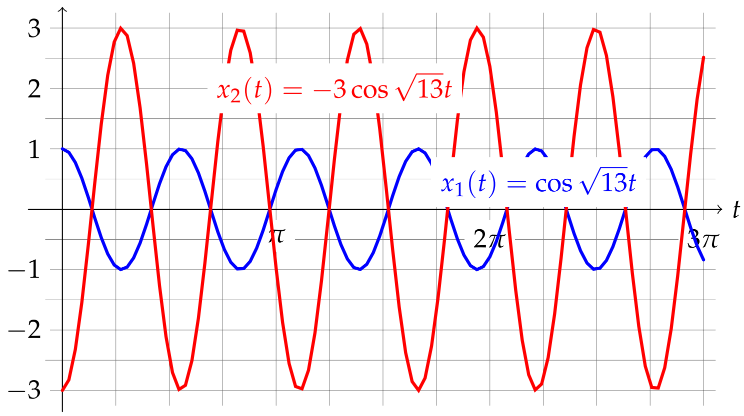
Is this periodic?

7. Suppose that an undamped, linear, time-invariant, two-degree-of-freedom mechanical system has a mode shape

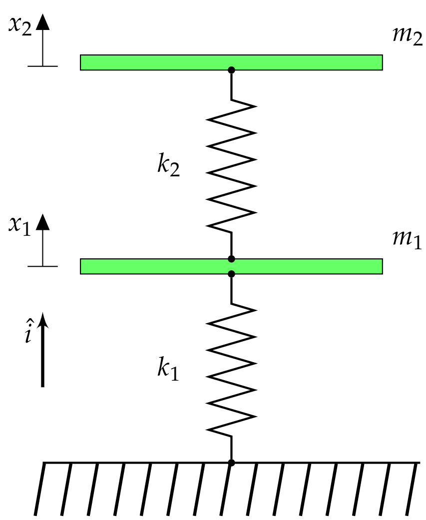
$$\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

with corresponding natural frequency ω. What is the ratio of the steady-state amplitudes of the two degrees of freedom if the inputs $f_1(t)$ and $f_2(t)$ are both harmonic with angular frequency ω and there is small damping present? What is the difference in phase shift between the two degrees of freedom?

8. Consider a linear, time-invariant, two-degree-of-freedom mechanical system that has two natural frequencies whose ratio is not a rational number. Suppose that a combination of impulses $f_1(t)$ and $f_2(t)$ results in the response shown in the graph below. Use this to determine one of the natural frequencies and the corresponding mode shape.



9. Consider the two-degree-of-freedom mechanical suspension, shown below, where x_1 and x_2 denote displacements of the lower and upper plate, respectively, relative to the undeformed configuration of the two springs.



Compute the natural frequencies, and comment on the determination of the corresponding mode shapes.

Solutions

1. Here, $\omega_n^2 = 4/2 \Rightarrow \omega_n = \sqrt{2}$.

2. Here, $x(t) = Ae^{-\gamma t}\cos(\omega t - \theta)$ implies that

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = -Ae^{-\gamma t}\big(\gamma\cos(\omega t - \theta) + \omega\sin(\omega t - \theta)\big)$$

and

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t) = Ae^{-\gamma t}\big((\gamma^2 - \omega^2)\cos(\omega t - \theta) + 2\omega\gamma\sin(\omega t - \theta)\big)$$

Substitution into the differential equation and identifying coefficients in front of $\cos(\omega t - \theta)$ and $\sin(\omega t - \theta)$ shows that

$$2(\gamma^2 - \omega^2) - 3\gamma + 4 = 0,\ 4\omega\gamma - 3\omega = 0$$

which imply that $\gamma = 3/4$ and $\omega = \sqrt{23}/4$.

Since $x(0) = A\cos\theta$ and $\mathrm{d}x/\mathrm{d}t(0) = -A(\gamma\cos\theta - \omega\sin\theta)$, it follows that

$$A\cos\theta = 1,\ A\sin\theta = -\frac{1}{\sqrt{23}} \Rightarrow A = \sqrt{\frac{24}{23}},\ \theta = -\arccos\sqrt{\frac{23}{24}}.$$

3. Here, $x_{ss}(t) = Ae^{\mathrm{j}\omega t}$, provided that

$$(-2\omega^2 + \mathrm{j}3\omega + 4)Ae^{\mathrm{j}\omega t} = e^{\mathrm{j}\omega t},$$

which implies that

$$A = \frac{1}{4 - 2\omega^2 + \mathrm{j}3\omega} = \frac{1}{\sqrt{(4-2\omega^2)^2 + 9\omega^2}}e^{\mathrm{j}\theta},$$

where

$$\cos\theta = \frac{4 - 2\omega^2}{\sqrt{(4-2\omega^2)^2 + 9\omega^2}},\ \sin\theta = -\frac{3\omega}{\sqrt{(4-2\omega^2)^2 + 9\omega^2}}.$$

4. Multiplication on both sides of,

$$\left(K - M\omega^2\right)\cdot\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

by M^{-1} implies that

$$\left(M^{-1} \cdot K - \omega^2 I\right) \cdot \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0,$$

i.e., that ω^2 is an eigenvalue of $M^{-1} \cdot K$ with corresponding eigenvector

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

5. Here, the natural frequencies ω satisfy the equation

$$0 = \det(K - M\omega^2) = 2\omega^4 - 5\omega^2 + 1$$

which implies that $\omega_{1,2} = \sqrt{5 \pm \sqrt{17}}/2$. The corresponding mode shapes satisfy the equation

$$0 = (K - M\omega^2) \cdot \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \frac{3 \mp \sqrt{17}}{4} A_1 - A_2 \\ -A_1 - \frac{3 \pm \sqrt{17}}{2} A_2 \end{pmatrix}$$

which imply that $A_2 = (3 \mp \sqrt{17}) A_1/4$.

6. The eigenvalues of the matrix

$$M^{-1} \cdot K = \begin{pmatrix} 4 & -6 \\ 0 & 7 \end{pmatrix}$$

are $\omega_1^2 = 7$ and $\omega_2^2 = 4$. The corresponding eigenvectors are

$$\begin{pmatrix} -2A \\ A \end{pmatrix} \text{ and } \begin{pmatrix} B \\ 0 \end{pmatrix}.$$

The function

$$x(t) = \begin{pmatrix} -2A \\ A \end{pmatrix} \cos(\sqrt{7}t - \theta_1) + \begin{pmatrix} B \\ 0 \end{pmatrix} \cos(2t - \theta_2)$$

satisfies the initial conditions provided that $\theta_1 = \theta_2 = 0$, $A = 2$, and $B = 5$. The solution is not periodic, since $\sqrt{7}/2$ is an irrational number.

7. If $f_1(t) = B_1 e^{j\omega t}$ and $f_2(t) = B_2 e^{j\omega t}$, then $x_1(t) = A_1 e^{j\omega t}$ and $x_2(t) = A_2 e^{j\omega t}$, where

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \approx \frac{k}{\epsilon} \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

for some constant k and $\epsilon \ll 1$. It follows that $|A_1|/|A_2| = 4/3$, and that $x_1(t)$ and $x_2(t)$ differ in phase by π, since $-1 = e^{j\pi}$.

8. Since $x_1(t)$ and $x_2(t)$ are harmonic, the response must be a free response described by one of the natural frequencies and its mode shape. Here, $\omega = \sqrt{13}$ and a mode shape is the column matrix

$$\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

The signals $x_1(t)$ and $x_2(t)$ differ in phase by π. It follows that when $x_1(t)$ has a local maximum, then $x_2(t)$ has a local minimum, and vice versa.

9. By Newton's 2nd law, the displacements $x_1(t)$ and $x_2(t)$ satisfy the differential equations

$$M \cdot \begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} + K \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $\dot{x}_i(t) = \mathrm{d}x_i(t)/\mathrm{d}t$, $\ddot{x}_i(t) = \mathrm{d}^2 x_i(t)/\mathrm{d}t^2$, and

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, K = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$$

It follows that the natural frequencies ω are the square roots of the eigenvalues of the matrix

$$M^{-1} \cdot K = \begin{pmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} \end{pmatrix}$$

Specifically,

$$\omega^2 = \frac{m_1 k_2 + m_2 k_1 + m_2 k_2 \pm \sqrt{(m_1 k_2 + m_2 k_1 + m_2 k_2)^2 - 4 m_1 m_2 k_1 k_2}}{2 m_1 m_2}.$$

The mode shapes are given by the corresponding eigenvectors.

Prelab Assignments

Complete these assignments before the lab. Show all work for credit.

1. Find the natural frequencies and mode shapes of a linear, time-invariant mechanical system with

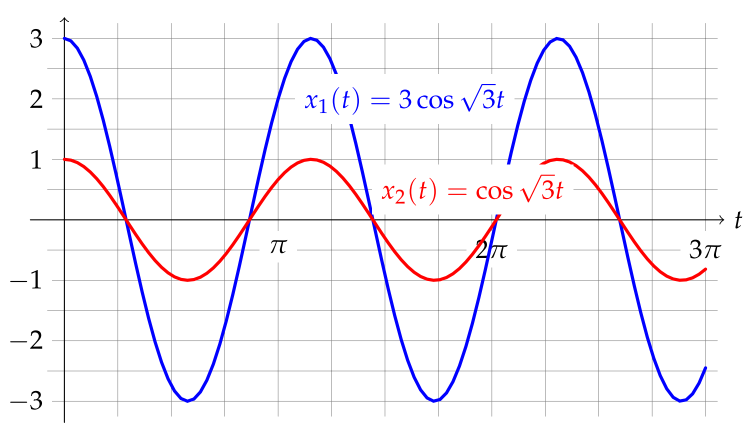
$$M = \begin{pmatrix} 3 & 0 \\ 0 & 3/4 \end{pmatrix}, K = \begin{pmatrix} 4 & -3 \\ -3 & 5 \end{pmatrix}.$$

2. Suppose that an undamped, linear, time-invariant, two-degree-of-freedom mechanical system has a mode shape

$$\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

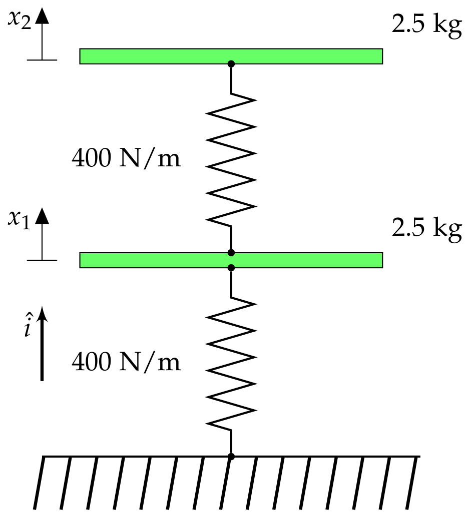
   with corresponding natural frequency 2. Sketch the steady-state response for the two degrees of freedom if the inputs $f_1(t)$ and $f_2(t)$ are both harmonic with frequency 1.98 and there is small damping present.

3. Consider a linear, time-invariant, two-degree-of-freedom mechanical system that has two natural frequencies whose ratio is not a rational number. Suppose that a combination of impulses $f_1(t)$ and $f_2(t)$ results in the response shown in the graph below. Use this to determine one of the natural frequencies and the corresponding mode shape.



4. Consider the two-degree-of-freedom mechanical suspension,

shown below, where x_1 and x_2 denote displacements of the lower and upper plate, respectively, relative to the undeformed configuration of the two springs.



Compute the natural frequencies and the corresponding mode shapes.

Lab instructions[3]

[3] These notes are an edited version of handouts authored by Andrew Alleyne.

In this lab, experiments will again be conducted on an Educational Control Products (ECP) Model 210 Rectilinear Dynamic System. This electromechanical apparatus is a three-degree of freedom spring-mass-damper system. It consists of configurable masses on low-friction bearings, springs, dampers and an input drive. Encoders connected to each of the carts determine their locations. The input drive command is generated by a digital to analog (DAC) signal produced by a data acquisition board installed in the benches' PC. The MATLAB toolbox REAL-TIME WINDOWS TARGET will be used to both drive the system when needed and collect response data.

Encoders attached to each cart are used to measure their displacements. These encoders have a resolution of 16,000 counts per revolution. The small pulley that attaches the cart to the encoder has a radius of 1.163 cm, so each cm of displacement is equivalent to $1/(2\pi * 1.163)$ revolutions and, consequently, $16,000/(2\pi * 1.163)$ encoder counts.

To make best use of lab time, make sure to note your observations for each plot as they are generated.

Impulse response

In the first experiment, you will use the impulse response of a two-degree-of-freedom system to determine the corresponding natural frequencies. Secure three large and one small brass weight on the carriage closest to the actuating motor and four large weights on the second carriage. Use two medium springs in the relevant places.

Experimental procedure

1. Turn on the equipment:
   (a) Start MATLAB using the icon found on your desktop.
   (b) Turn on the ECP control box on the top shelf.

(c) Inside MATLAB, open the "read-only" file `twoDOFImpulse.mdl` located in the directory `N:\labs\me340\Mass_Spring_twoDOF`.

(d) Save the file with the name `lab5free<yourNetID>.mdl` in the directory `C:\matlab\me340`.

(e) Change MATLAB's current directory to the location where your model file was saved by typing `cd c:\matlab\me340` at the MATLAB command prompt.

(f) Click the SIMULINK model file to give it focus. Confirm that the gain block is converting encoder counts to distance in cm and that data is sampled by Real-Time Windows Target every 5 ms. Confirm that the Pulse generator is producing a 6 sample period pulse ($6 * 0.005 = 0.030$ s pulse) with an amplitude of 5 DAC V.

(g) Enter <ctrl+B> on your keyboard to build the model. While your auto-generated code is building, watch the status at the bottom left of your SIMULINK model to determine when the build is complete.

2. Acquire data:

(a) Simply press the green Run button to start data collection. Note, it can take a number of seconds to start the acquisition so don't click the run button multiple times if the data collection does not start right away.

(b) Click on the "Cart1_Pos" plot to give it focus and see if you can visually identify two different frequencies in the system response. The higher frequency is easier to spot then the lower one.

3. Analyze data:

(a) In MATLAB, perform spectral analysis using the fast fourier transform `fft` command to decompose the noisy signal into its harmonic components.

```
>> pos1 = twoDOFImpulseCart1_data(:,2);
```

```
>> pos1fft = fft(pos1, 1024);
>> Pd = pos1fft.*conj(pos1fft)/1024;
>> T=.005;
>> freq=(1/T)/1024*(0:511);
>> plot(freq, Pd(1:512))
```

(b) Zoom into the plot to determine the frequencies corresponding to peaks in the power spectral density and record these. Consider the effect of the resolution on the horizontal axis and identify an associated bound on the frequency error.

(c) Use the `axis` command to set the range of frequencies and power spectral density values. Use the `title` and `xlabel` commands to add the title "Power Spectral Density Plot" and the axis label "Frequency (Hz)" to the horizontal axis.

Harmonic Input Forced Response

As an alternative to using the impulse response, in this experiment you will use the response to a sine wave input with slowly increasing frequency, but constant amplitude, to observe changes to the amplitude and phase associated with resonance at the system's natural frequencies.

Experimental procedure

1. Turn on the equipment:

   (a) Close the SIMULINK model that you used in the previous experiment.

   (b) Open the "read-only" file `twoDOFsweptsine.mdl` located in the directory `N:\labs\me340\Mass_Spring_twoDOF`.

   (c) Save the file with the name `lab5sweep<yourNetID>.mdl` in the directory `C:\matlab\me340`

   (d) Change MATLAB's current directory to the location where you model file was saved by typing `cd c:\matlab\me340` at the MATLAB command prompt.

(e) Click the SIMULINK model to give it focus. In this model, we use a *Function* component to vary the input to cart one's motor. The desired input is a range of frequencies that start at 0.625 Hz and increase over a time period of 150 seconds to 5 Hz. Confirm that the output of the *Function* component is the signal $0.4 \sin\left(2\pi\left(0.625t + \frac{5.0-0.625}{150}\frac{t^2}{2}\right)\right)$ DAC V.

(f) Enter <ctrl+B> on your keyboard to build the model.

2. Acquire data:

(a) Simply press the green Run button to start data collection. Note, it can take a number of seconds to start the acquisition so don't click the run button multiple times if the data collection does not start right away. You will note that in the "Cart1" plot window three things are being plotted: cart response, input waveform, and a straight line that corresponds to the current frequency of the sine sweep waveform. As the sine sweep progresses you should notice two resonance frequencies.

(b) When you have observed both resonance frequencies, click the Stop button to stop data collection.

3. Analyze data:

(a) In MATLAB, create a plot of the response versus the excitation frequency

```
>> data = twoDOFsweptsineCart1_data;
>> plot(data(:,2), data(:,4))
>> xlabel('Frequency (Hz)'); ylable('Amplitude (cm)');
```

(b) Zoom in on your plot and record the resonance frequencies. Compare these with the modal frequencies obtained in Experiment I.

Mode shapes

Recall that mode shapes are described by the relative amplitudes and the phase difference between the displacements of the two degrees of freedom when they oscillate at the natural frequencies. In the case of small damping, we can typically determine the mode shape experimentally by exciting the system with a harmonic input near a natural frequency. This is the approach used in this experiment.

Experimental procedure

1. Initialize experiment:

   (a) Close the Simulink file that you used in the previous experiment.

   (b) Open the "read-only" file `twoDOFmodeshape.mdl` located in the directory `N:\labs\me340\Mass_Spring_twoDOF`.

   (c) Save the file with the name `lab5mode<yourNetID>.mdl` in the directory `C:\matlab\me340`.

   (d) Change MATLAB's current directory to the location where you model file was saved by typing `cd c:\matlab\me340` at the MATLAB command prompt.

   (e) Click the SIMULINK model to give it focus. Enter <ctrl+B> on your keyboard to build the model.

   For each of the two natural frequencies found in the previous experiments:

2. Aquire data:

   (a) Enter the natural frequency into the *Sine wave* component, and click the Run button to start data collection. Let the system run for 10 seconds or so to allow time for it to reach steady state. Note that the yellow trace is cart one and the purple trace is cart two.

   (b) After steady state has been reached and enough data has been collected, click the Stop button to stop data collection.

3. Analyze data:

   (a) In MATLAB, plot the displacements of the two masses:

```
>> data = twoDOFmodeshapeCarts_data;
>> plot(data(:,1), data(:,2),'r-', ...
     data(:,1), data(:,3) ,'b-.')
```

(b) Zoom in on your plot and measure the amplitude ratio and phase difference between the signals.

Report Assignments

Complete these assignments during the lab. *Show all work for credit.*

1. Write down the natural frequencies found from the power spectral density plot of the free response in the first experiment. Don't forget units.

2. Write down the resonance frequencies found from the analysis of the harmonically forced response in the second experiment. Don't forget units.

3. Compare the frequencies obtained in the previous two cases to the values found in the last prelab assignment.

4. For each of the resonance frequencies, identify the forced response in the third experiment in terms of the observed ratio of amplitudes and phase difference between the two degrees of freedom.

5. Compare the observed ratios of amplitudes and phase differences to the mode shapes found in the last prelab assignment.