

### Laboratory handout 6 – Continuous systems

The lateral displacement  $y(x, t)$  of a homogeneous, slender beam of density  $\rho$  per unit length that results from the imposition of a transversal load  $f(x, t)$  per unit length and suitable boundary conditions at the ends  $x = 0$  and  $x = \ell$  is governed by the **partial differential equation**

$$\rho \frac{\partial^2 y}{\partial t^2}(x, t) + c \frac{\partial^5 y}{\partial x^4 \partial t}(x, t) + EI \frac{\partial^4 y}{\partial x^4}(x, t) = f(x, t), \quad (1)$$

where  $E$  is the **Young's modulus** of the beam,  $I$  is the **area moment of inertia** of a cross section, and  $c$  is a **damping coefficient**.

For a beam clamped at both ends, the boundary conditions imply that

$$y(0, t) = 0, \quad \frac{\partial y}{\partial x}(0, t) = 0, \quad y(\ell, t) = 0, \quad \frac{\partial y}{\partial x}(\ell, t) = 0. \quad (2)$$

Similarly, for a cantilevered beam with one clamped end and one free end, the boundary conditions imply that

$$y(0, t) = 0, \quad \frac{\partial y}{\partial x}(0, t) = 0, \quad \frac{\partial^2 y}{\partial x^2}(\ell, t) = 0, \quad \frac{\partial^3 y}{\partial x^3}(\ell, t) = 0. \quad (3)$$

When  $c = 0$ , the **free response** (with  $f(x, t) = 0$  for  $x \in [0, \ell]$  and  $t \geq 0$ ) is of the form  $\gamma(x)e^{j\omega t}$  for some function  $\gamma(x)$  and angular frequency  $\omega$  provided that

$$\gamma''''(x) = \frac{\rho\omega^2}{EI} \gamma(x), \quad (4)$$

where  $\gamma(x)$  must satisfy the boundary conditions at  $x = 0$  and  $x = \ell$ .

If a nonzero function  $\gamma(x)$  satisfies (4) and the boundary conditions for some  $\omega$ , then  $\omega$  is a **natural frequency** of the beam and  $\gamma(x)$  is the corresponding **mode shape**. The expression  $\gamma(x)e^{j\omega t}$  is a **modal oscillation**.

For a beam clamped at both ends, every mode shape must satisfy

the boundary conditions

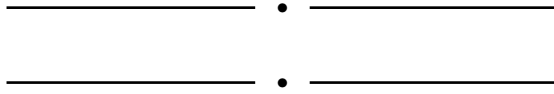
$$\gamma(0) = \gamma(\ell) = \gamma'(0) = \gamma'(\ell) = 0. \quad (5)$$

Similarly, for a cantilevered beam with one clamped end and one free end, every mode shape must satisfy the boundary conditions

$$\gamma(0) = \gamma'(0) = \gamma''(\ell) = \gamma'''(\ell) = 0. \quad (6)$$

For either choice of boundary conditions, there exists an infinite, increasing sequence of distinct natural frequencies  $\{\omega_n\}_{n=1}^{\infty}$  and associated mode shapes  $\gamma_n(x)$ , such that

$$\int_0^\ell \gamma_n^2(x) dx = 1. \quad (7)$$



For either choice of boundary conditions, every solution to (1) can be written as a **modal decomposition**

$$y(x, t) = \sum_{n=1}^{\infty} a_n(t) \gamma_n(x) \quad (8)$$

for some unknown functions  $a_n(t)$ . Substitution into (1) then yields

$$\sum_{n=1}^{\infty} \left( \rho \ddot{a}_n(t) \gamma_n(x) + c \dot{a}_n(t) \gamma_n^{iv}(x) + EI a_n(t) \gamma_n^{iv}(x) \right) = f(x, t) \quad (9)$$

Using (4), (7), and the orthogonality of mode shapes, multiplication by  $\gamma_m(x)$ , integration, and division by  $\rho$  yields

$$\ddot{a}_m(t) + \frac{c\omega_m^2}{EI} \dot{a}_m(t) + \omega_m^2 a_m(t) = \frac{1}{\rho} \int_0^\ell f(x, t) \gamma_m(x) dx. \quad (10)$$

For each  $m$ , this corresponds to a single-degree-of-freedom oscillator with damping ratio  $\zeta = c\omega_m/2EI$  and natural frequency  $\omega_m$ .

For small but nonzero  $\zeta$  and

$$\frac{1}{\rho} \int_0^\ell f(x, t) \gamma_m(x) dx = B_m e^{j\omega t} \quad (11)$$

with excitation amplitudes  $B_n$  and excitation frequency  $\omega$ , it follows that

$$y_{ss}(x, t) \approx \frac{B_n}{j2\zeta\omega_n} e^{j\omega t} \gamma_n(x) \quad (12)$$

for  $\omega \approx \omega_n$ . In other words, at the resonance frequencies  $\omega_n$ , large-in-magnitude steady-state amplitude results even for small-in-magnitude excitation amplitude. At these frequencies, the spatial shape of the response is identical to a mode shape.

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### Exercises

1. Show that there is an infinite, increasing sequence of natural frequencies for the beam with both ends clamped, and find expressions for the corresponding mode shapes.
2. For the beam with both ends clamped, show that

$$\frac{\rho\omega_n^2\ell^4}{EI} \approx \pi^4 \left(n + \frac{1}{2}\right)^4.$$


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### Solutions

1. If (4) has a solution  $\gamma(x)$  that satisfies the boundary conditions and does not equal 0 for all  $x$  then  $\omega$  is the corresponding natural frequency of the beam. The general solution to (4) is of the form<sup>1</sup>

$$\gamma(x) = C_1 \cos \lambda \frac{x}{\ell} + C_2 \sin \lambda \frac{x}{\ell} + C_3 \cosh \lambda \frac{x}{\ell} + C_4 \sinh \lambda \frac{x}{\ell},$$

where  $\lambda = (\rho\omega^2/EI)^{\frac{1}{4}} \ell$  and  $C_1, C_2, C_3$ , and  $C_4$  are arbitrary. The first three boundary conditions in (5) are satisfied provided that  $C_3 = -C_1$ ,  $C_4 = -C_2$  and

$$C_2 = \frac{\cosh \lambda - \cos \lambda}{\sin \lambda - \sinh \lambda} C_1.$$

<sup>1</sup> Remember that

$$\cos x = \frac{1}{2}(e^{jx} + e^{-jx}),$$

$$\sin x = \frac{1}{j2}(e^{jx} - e^{-jx}),$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}),$$

and

$$\sinh x = \frac{1}{2}(e^x - e^{-x}).$$

This implies that if  $\omega$  is a natural frequency, then the corresponding mode shape is a nonzero multiple of

$$\cos \lambda \frac{x}{\ell} - \cosh \lambda \frac{x}{\ell} + \frac{\cosh \lambda - \cos \lambda}{\sin \lambda - \sinh \lambda} \left( \sin \lambda \frac{x}{\ell} - \sinh \lambda \frac{x}{\ell} \right).$$

The final boundary condition requires that

$$\frac{2\lambda(\cos \lambda \cosh \lambda - 1)}{\ell(\sin \lambda - \sinh \lambda)} = 0$$

or, in other words,

$$\cos \lambda = \frac{1}{\cosh \lambda}.$$

The left-hand side oscillates between  $-1$  and  $1$  as  $\lambda$  increases, while the right-hand side is positive and decreases monotonically from  $1$  at  $\lambda = 0$  to  $0$  as  $\lambda \rightarrow \infty$ . It follows that there is an infinite, increasing sequence  $\lambda_1, \dots, \lambda_n, \dots$  of positive real values for which the two sides are equal and, consequently, an infinite, increasing sequence of natural frequencies  $\{\omega_n\}_{n=1}^{\infty}$ , with corresponding mode shapes  $\gamma_n(x)$  that are multiples of the function

$$\cos \lambda_n z - \cosh \lambda_n z + \frac{\cosh \lambda_n - \cos \lambda_n}{\sin \lambda_n - \sinh \lambda_n} (\sin \lambda_n z - \sinh \lambda_n z).$$

where  $z = x/\ell$ .

2. As seen in the previous solution, the parameter combination  $\rho\omega_n^2\ell^4/EI$  corresponds to the fourth power of the  $n$ th root of the equation

$$\cos \lambda = \frac{1}{\cosh \lambda}.$$

A graph of the two sides shows that the roots occur close to the zeros of  $\cos \lambda$ , i.e.,  $\pi/2 + n\pi$ . In particular, for the lowest natural frequency with  $n = 1$ ,

$$\omega_1 \approx 22\sqrt{\frac{EI}{\rho\ell^4}}.$$

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## Prelab Assignments

Complete these assignments before the lab. Show all work for credit.

1. The mode shapes  $\gamma_n(x)$  of a cantilever beam are multiples of the function

$$\sinh \lambda_n z - \sin \lambda_n z - \frac{\sinh \lambda_n + \sin \lambda_n}{\cosh \lambda_n + \cos \lambda_n} (\cosh \lambda_n z - \cos \lambda_n z)$$

where  $z = x/l$  and  $\lambda_n$  are roots of the equation

$$\cos \lambda = -\frac{1}{\cosh \lambda}.$$

Plot the first three mode shapes of the cantilever beam. (Hint: Compute the first three roots of  $\lambda$  and substitute in the mode shape relation).

2. For a given mode shape, a **node** along the beam is a location where the displacement  $y(x, t) = \gamma(x)e^{j\omega t} = 0$  for all  $t$ . Determine the number of nodes for the first two natural frequencies of the cantilevered beam. How many nodes do you expect for the  $n$ th mode shape?
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## Lab instructions<sup>2</sup>

<sup>2</sup> These notes are an edited version of handouts authored by Andrew Alleyne.

In previous experiments with the ECP rectilinear plant, we studied systems where the flexibility and mass could be separated. This is called a lumped system since flexibility and inertia were “lumped” into the individual elements (spring and mass). Lumping is a method of approximating the real system to obtain a simplified model. We were able to lump the ECP system into mass and spring because the flexibility of the cart was insignificant when compared to the spring, and the spring’s mass was insignificant when compared to that of the cart. To simplify the model, we assumed that the cart was rigid and that the spring had no mass.

What can be done when the mass does have significant flexibility and the spring does have a significant mass? In such a case, the mass and the spring are called continuous systems. In continuous systems, it is very difficult or impossible to “lump” the system parameters. In other words, mass and flexibility are distributed throughout the system, not at discrete points.

Engineering applications are rarely as simple as a mass-spring system. In many cases there will be components that have significant amounts of inertia and flexibility. This is true of almost any structure, from a high-rise building to the wing of an aircraft. In the high-rise building, the steel frame acts as a spring and a mass. Other parts of the building, e.g., concrete floors and walls, act as masses that are distributed throughout the structure. It may not be easy to separate the components into a mass, spring, and damper!

In this lab, we will perform experiments on a cantilevered beam with one end clamped and one end free. Such a beam has an infinite, increasing sequence of natural frequencies  $\{\omega_n\}_{n=1}^{\infty}$  with corresponding mode shapes  $\gamma_n(x)$ , such that  $\gamma_n(x)e^{j\omega_n t}$  is an example free response.

Moreover, in the presence of light damping, a resonant, large-amplitude response results from harmonic excitation with excitation frequency near a natural frequency. Usually, a system will experience

excitation only in a certain range of frequencies, and we need only study/model those modes whose frequencies lie in this range.

### *Experiment I: Frequency response*

We will perform a frequency response experiment on an actual beam to determine precise values of the first three modal frequencies and sketch the corresponding mode shapes.

A small linear motor is used to exert force on the beam. A 600-watt audio amplifier powers the exciter. The output voltage of the amplifier, and thus the force of the exciter, will be determined by a sinusoidal signal from an HP 33120A Function Generator. To measure the motion of the beam, we will use a small accelerometer attached to the free end of the beam. The signal from the accelerometer will be passed through a signal-conditioning amplifier and then to an oscilloscope. We will use the measurement capabilities of the oscilloscope to record acceleration amplitudes for our frequency response experiment. Finally, we will use a strobe light to observe the mode shapes of the beam.

#### **Important Notes on the Oscilloscope:**

- If the oscilloscope trace becomes too large to fit on the screen, turn the CH2 VOLTS/DIV knob until the whole trace is visible.
- The oscilloscope will not automatically measure peak-to-peak voltage if the signal amplitude goes off-screen.
- At higher frequencies, you will need to turn the SEC/DIV knob to look at smaller segments of data. You can also try pushing the MAIN/DELAYED button and picking MAIN or ROLL from the screen menu.
- The oscilloscope's automatic peak-to-peak measurement is based on counting pixels. If your signal is too small on screen, the resolution of your measurement will be poor! Try to keep the signal on the majority of the screen.

Before beginning the experiment, make sure all nuts and bolts on the experimental apparatus are tight. *Make sure to do the following steps in order so as to avoid damaging the equipment!*

### *Experimental procedure*

#### 1. Turn on the equipment:

- (a) Turn on the oscilloscope.
- (b) Turn on the accelerometer amplifier (small box with meter).  
Set the gain to 10.
- (c) Turn on the function generator.
  - i. Place the function generator in “High Z” mode by pressing the following button sequence: SHIFT, MENU,  $\rightarrow\rightarrow\rightarrow$ ,  $\downarrow\downarrow$ ,  $\rightarrow$ . You should now see *High Z* on the display. Press ENTER.
  - ii. Set the output amplitude. Press the AMPL button followed by the ENTER NUMBER button. Use the number keys to input 0.100 and then press the VPP button. You have now set the amplitude of excitation to 100 mV peak-to-peak (mV p-p). *Verify that this correct number shows on the display. Using a higher level will damage the system.*
  - iii. Set the output frequency. Press the FREQ button then the ENTER NUMBER button. Use the number keys to input 1.00. Press the HZ button. You have now set the frequency of excitation to 1.00 Hz.
- (d) On the oscilloscope, you should see two traces. Channel 1 should be the smooth sine wave from the function generator. Channel 2 should be the signal from the accelerometer, which is a fuzzy flat line for now. Tap the beam to verify that the accelerometer is working – the Channel 2 signal should spike as a result.
- (e) Turn on the Hafler power amplifier. The beam should start to vibrate slightly at 1 Hz.



## 2. Acquire data:

- (a) For each of the frequencies in the data table in the Report Assignments section, record the peak-to-peak input amplitude  $V_{in}$  and accelerometer voltage  $V_{acc}$ .
- (b) Specifically, measure the peak-to-peak amplitude of the accelerometer voltage by doing the following on the oscilloscope:
  - i. Push the VOLTAGE button in the *measure* area.
  - ii. Push the screen menu button labeled SOURCE to select Channel 2.
  - iii. Push the screen menu button  $v_{P-P}$ . The peak-to-peak accelerometer voltage will appear on the bottom of the oscilloscope screen as  $V_{p-p}(2)$ .
- (c) Increase the frequency to the next value on the table by pushing the FREQ button on the function generator. Use the  $\leftarrow$  and  $\rightarrow$  keys to pick the digit you wish to change, then turning the large knob to increase or decrease the digit. Allow the accelerometer signal to settle down to a steady sine wave after each change of frequency before recording the accelerometer peak-to-peak voltage from the oscilloscope.
- (d) Looking at your data table, you should notice three frequencies at which there are local resonance peaks in the accelerometer response. Reset the excitation frequency to each of these values and scan slowly until you precisely (within 0.1 Hz) locate the frequencies at which these maxima occur. Note these frequencies in the space provided underneath the data table. Also record the peak-to-peak acceleration voltage  $V_{acc}$  for these.
- (e) As was stated before, this beam has an infinite number of natural frequencies. To verify this, slowly turn the excitation frequency beyond the maximum on the data table. As you turn the frequency higher and higher, you will observe (and hear!) many more peaks in the vibration, each corresponding to one of the higher natural frequencies of the beam.

- (f) Turn off the equipment in the following order to avoid equipment damage!
  - i. Hafler power amplifier
  - ii. Function generator
  - iii. Accelerometer amplifier (make sure to do this)
  - iv. Oscilloscope

3. Analyze data:

- (a) Create a Bode diagram of the magnitude of the frequency response function in dB. Specifically, for each excitation frequency, let the corresponding  $y$ -coordinate equal  $20 \log_{10} (V_{acc} / V_{in})$ . Use the `MATLAB` `semilogx` command to represent the excitation frequency using a logarithmic scale along the horizontal axis.
- (b) Once everyone has completed the above portions of the experiment, your TA will use a strobe light to illuminate one of the beams. The TA will set the excitation frequency to each of the natural frequencies found above, and use the strobe light to “freeze” the motion of the beam. Sketch the shape of the frozen beam at each natural frequency. This shape is the mode shape for that natural frequency.

### *Experiment II: Finite-element modeling*

In practice, the material and geometric properties of beam structures are not uniform, but vary along the structure. For such structures, it is difficult, if not impossible, to obtain exact solutions for the natural frequencies and mode shapes. As an alternative, since the 1950s, engineers have been using a numerical method called Finite Element Analysis (FEA). This method is computationally intensive but has become very widely used as computing costs have fallen. FEA is used to solve many engineering problems including stress analysis, vibration analysis, heat transfer, and electromagnetic field analysis.

The details of FEA are beyond the scope of this class, but the method basically consists of breaking our beam into small pieces

(called *elements*) that are joined together at points called *nodes*. Each element has uniform stiffness and mass along its length and can thus be simply modeled. The interaction of each element with its neighbors is taken into account, and global mass and stiffness matrices can be produced for the system. As with the two-DOF mass-spring system in Lab 5, the mass and stiffness matrices can be used to solve for the natural frequencies and mode shapes. If the beam is broken into sufficiently small elements, the finite element solution approximates the experimental result closely.

#### *Experimental procedure*

1. Start MATLAB on your PC.
2. Type `cd n:\labs\me340\beam` and then type `beam` to run the m-file `beam.m`. This particular program will break the beam into only 3 elements. MATLAB will use FEA to find the first three natural frequencies and approximations of mode shapes for the experimental beam. Print out the three plots that appear.

## Report Assignments

Complete these assignments during the lab. *Show all work for credit.*

1. Enter the data from Experiment I in the following table.

Freq (Hz)	$V_{acc}$ p-p	$V_{in}$ p-p
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		

Freq (Hz)	$V_{acc}$ p-p	$V_{in}$ p-p
15		
16		
17		
18		
19		
20		
25		
30		
35		
40		
45		
50		
55		
60		

2. Record the first three natural frequencies within 0.1 Hz, as well as the corresponding peak-to-peak accelerator voltage.

$$\omega_1 = \quad \quad \quad V_{acc} =$$

$$\omega_2 = \quad \quad \quad V_{acc} =$$

$$\omega_3 = \quad \quad \quad V_{acc} =$$

3. Attach the Bode diagram of the magnitude of the beam's frequency response function.
4. Attach a sketch of the first three mode shapes of the beam.
5. Attach the three plots generated by running the FEA in MATLAB.
6. Compare the FEA mode shapes to the sketched experimental mode shapes. Do the shapes roughly match? What could be done

to make the FEA mode shapes more like what you saw in the experiment?

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