ME 360: FUNDAMENTALS OF SIGNAL PROCESSING, INSTRUMENTATION AND CONTROL

Experiment No. 4
Modeling and Identification of an Electric Motor using Step Response Methods
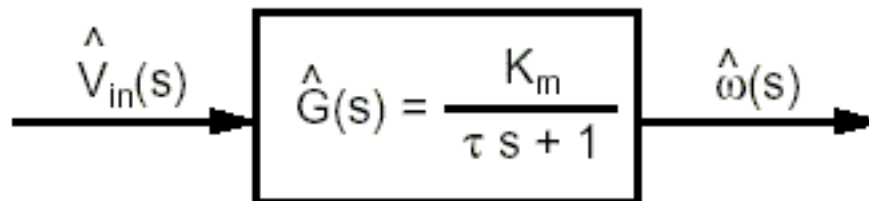
1. CREDITS

Originated: T-C. Tsao and N. R. Miller, October 1995
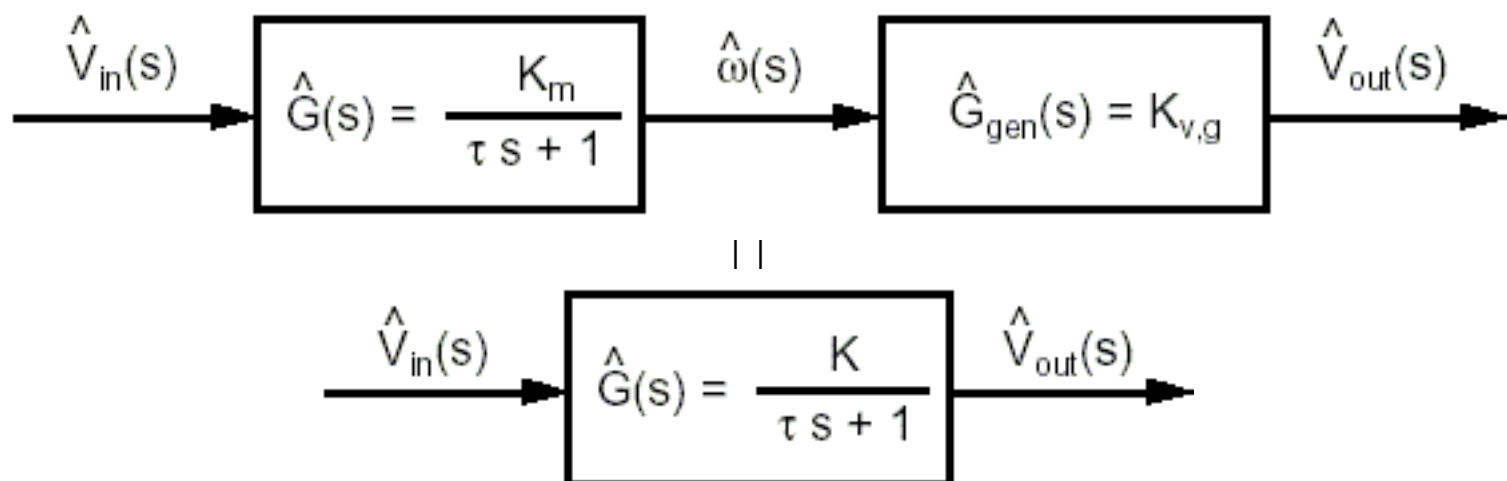Last Updated: D. Block, October 2018

2. OBJECTIVE

Measure the steady-state gain K and time constant τ of a motor-generator system.

3. KEY CONCEPTS

(a) A DC motor may be approximated as a first-order, linear system where the system input is the applied voltage $V_{in}(t)$ and the system output is motor speed $\omega(t)$.

$$\hat{V}_{in}(s) \longrightarrow \boxed{\hat{G}(s) = \frac{K_m}{\tau s + 1}} \longrightarrow \hat{\omega}(s)$$

(b) In this experiment, a motor-generator combination serves as a first-order system with a voltage input V_{in} and a voltage output V_{out}.

$$\hat{V}_{in}(s) \longrightarrow \boxed{\hat{G}(s) = \frac{K_m}{\tau s + 1}} \xrightarrow{\hat{\omega}(s)} \boxed{\hat{G}_{gen}(s) = K_{v,g}} \longrightarrow \hat{V}_{out}(s)$$

$$||$$

$$\hat{V}_{in}(s) \longrightarrow \boxed{\hat{G}(s) = \frac{K}{\tau s + 1}} \longrightarrow \hat{V}_{out}(s)$$

(c) The two parameters characterizing a first-order system are the *steady-state gain* K and the *time constant* τ.
(d) A linear model can be used even though K varies with input voltage and the system is actually nonlinear. Models are rarely perfect.
(e) We determine K and τ by observing the system response to a known input.
(f) *System response* is the time history of the system output.
(g) Determining system parameters from measured response data is known as *system identification*.
(h) Several alternative methods exist for extracting K and τ from the system-response data. These methods differ in terms of
   (i)   the number of data points used (from a single value to all the data) and
   (ii) graphical vs. numerical

(i) Human evaluation and judgment can be a useful component of the data analysis and yield valuable insights into dynamic system behavior.

4. OVERVIEW OF METHODS

In this experiment, we study the steady-state and dynamic characteristics of a DC motor coupled to a DC generator. The DC generator is used as a tachometer to measure the speed of the motor. The motor-generator combination is modeled as a first-order, linear system with steady-state gain K and time constant τ. A first-order, linear system is the simplest type of dynamic system and thus provides a good introduction to important dynamic system concepts. The first-order system is also important from a practical viewpoint as many real-world systems can be modeled in this manner. The DC motor considered in this experiment is just one such example.

The motor-generator system considered here is of the basic input-output type. The single input is V_{in}, the voltage applied to motor. The single output is V_{out}, the voltage produced by the generator. This first-order system may be described in the following ways.

Governing Equation: $\tau \dfrac{dV_{out}}{dt}(t) + V_{out}(t) = K V_{in}(t)$ $\qquad\qquad V_{out}(0) = V_o$

Transfer Function: $\hat{G}(s) = \dfrac{K}{\tau s + 1}$

Steady-state Response: $V_{out}(t) = K V_{in}$ $\qquad\qquad V_{in}(t) = $ constant, all t

Step Response: $V_{out}(t) = K (V_o - V_{in}) \exp(-t / \tau) + K V_{in}$ $\qquad\qquad V_{in}(t) = $ constant, $t \geq 0$

Our goal is to determine K and τ and compare these measurements with values calculated from a more detailed physical model of the motor-generator system. This more detailed model relates K and τ to other physical parameters that can be independently measured. Based on nominal values reported by the manufacturer, we conclude that **K lies between 1.1 and 1.3 and τ lies between 40 ms and 80 ms**.

There are many ways to obtain K and τ from observed system behavior. This diversity is good because it puts a wide variety of useful tools at our disposal. This diversity is bad because we must sort out the differences between the various methods and assess the relative merits of each. Our choice is complicated by two real-world problems:

(a) Our model of the system is imperfect.
(b) Our methods of measuring system response are imperfect.

Modeling errors arise from system nonlinearities and higher-order effects; experimental errors arise from noise and measurement uncertainty. Thus, different methods may give different values of K and τ. We wish to identify those methods that are least sensitive to modeling and experimental errors. Our basic strategy here is that the more data used in the analysis, the better the overall result will be. This strategy leads us away from local or point methods and towards global or integral methods. Averaging is usually an effective means of reducing error in measured values.

5. SYNOPSIS OF PROCEDURE

The digital multimeter, function generator, PC-based hardware, and MATLAB software are used to carry out the following measurements.

(a) The *steady-state gain* of the system is determined by applying a series of input voltages to the motor and measuring the resulting steady-state output voltages of the generator using the digital multimeter. Here, we see that K varies with input voltage indicating that the system is slightly nonlinear.
(b) The *step response* of the system is studied by making a step change in the input voltage to the motor and sampling the output voltage of the generator as a function of time using the oscilloscope and PC-based data-acquisition system. The system parameters K and τ are then determined from the sampled output voltage waveform using the following four methods.

    (i) Locate the time at which the voltage change is 63.2 % of the maximum change.
    (ii) Find the intersection of the tangent at t = 0 with the steady-state asymptote (a horizontal line at the final voltage).
    (iii) Use a numerical method based on the integral of the step response data.
    (iv) Plot the observed response together with the predicted response based on guessed values of K and τ; iterate on K and τ until good agreement is achieved.

6.   PROCEDURE

Important General Information – Please Read Carefully

(a)   ***Always turn off the power supplies when changing connections. Dangling leads can easily contact the metal tabletop creating a short, blowing a fuse, creating an unsafe situation or damaging the equipment.***

(b)   *Disconnect the leads from the instruments when not in use. Connect the instruments last after the wiring is carefully checked.*

(c)   *If your station is missing something, ask your Laboratory Assistant to replace it. Do not take items from other stations.*

6.1 Wiring the System

Figure 1 shows how the system is wired.

Detailed Information for Wiring System

A brief summary of the connections is given below.

DAC Output to Power Amplifier
A shielded cable with three banana plugs on each end connects Analog Output Channel 0 to the input of the Power Amplifier (labeled simply "Amplifier" on the patch panel). Red is positive or high, black is negative or low, and white is the shield. At the amplifier end, connect red to red, black to black, and white to white. At the DAC end, connect red to red and both black and white to black.

Amplifier Output to Motor Input
A second shielded cable with three banana plugs on each end connects the output of the power amplifier to the input of the motor. Red is positive or high, black is negative or low, and white is ground. Follow red-black-white color-coding on each end of the cable.

Generator Output to ADC Input
Two banana-plug patch cords connect the generator output to Analog Input Channel 0. On the generator end, orange is positive or high and gray is negative or low. On the analog-input end, red is positive or high and black is negative or low.

Connections to Oscilloscope
The DAC output is routed to Channel 1 of the oscilloscope, and the generator output is routed to Channel 2 so that the system input and output waveforms can both be displayed. Two cables each with a BNC connector on one end and a pair of banana plugs on the other end are used for this purpose. Red-black color-coding is followed.

Connections to Digital Multimeter
A pair of banana-plug patch cords connects the inputs of Analog Channel 0 to the voltage inputs of the multimeter. Red-black color-coding is again observed.

Push Button
The push button is connected to the phone jack on the patch panel in the section labeled "Amp Inhibit".
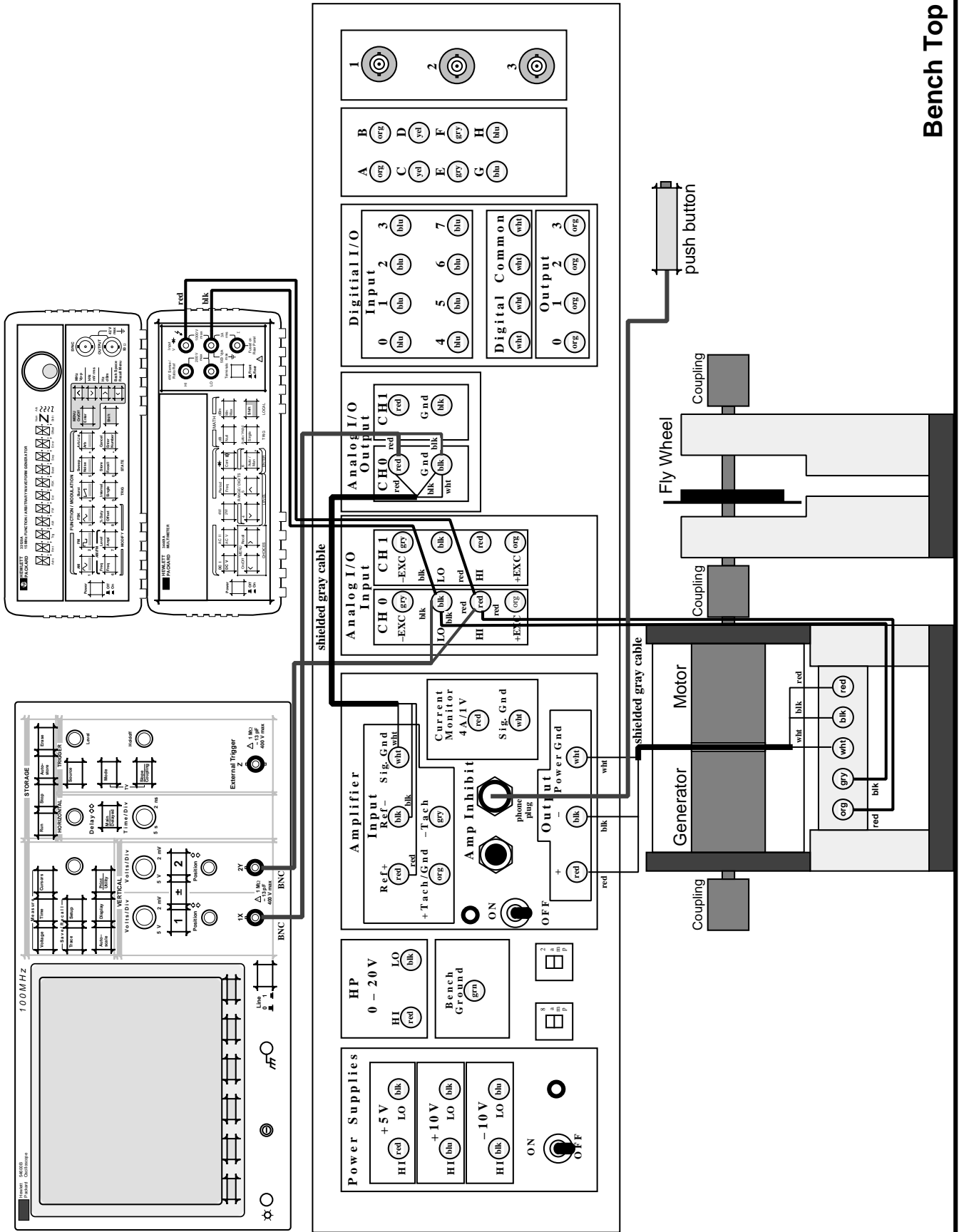
Figure 1. Wiring diagram and equipment layout for motor-generator system experiment – motor input from analog output.

6.2 Steady-state Gain

6.2.1 Initial Startup

(a) Power on the oscilloscope and digital multimeter. Recall the default oscilloscope settings and display both Channels 1 and 2. Change the time mode to "roll" by pressing the "Main/Delayed" button and selecting the correct setting. Set the vertical scale to 5 V/div for Channel 1 and 1 V/div for Channel 2. Set the horizontal scale to 500 ms/div. Move the reference for Channel 2 to the bottom of the screen and the reference position for Channel 1 to the second grid division from the top.

(b) Start MATLAB.

6.2.2 Determine Steady-state Gain for Motor Inputs of 3, 4, 5 and 6 V

(c) We will use a MATLAB function 'meu2y' to collect data from the benches PC analog inputs. See Appendix D for more help on the 'meu2y' MATLAB function.

(d) Use the commands given below to generate a constant input to the motor for 10 s. After steady state is reached, measure the output of the generator using the DMM. Enter this value on the Data Sheet. Compute the gain by taking the ratio of the output voltage to the input voltage.

MATLAB Command	Action
`vdacs = [3 4 5 6]`	enter the test input voltages into an array
`tstep = 0.010`	set a 10-ms time step
`runtime = 10`	set a run time of 10 s
`t = tstep:tstep:runtime`	set up the array of sampling times from 0 to 10 s in 10-ms steps; `t` is thus a *row* vector of length 1000 equal to `[0.01 0.02 0.03 ... 9.99 10.00]`
`t = t'`	convert `t` to an equivalent *column* vector
`vdac = vdacs(1) * ones(length(t),1);`	generate a *column* vector of length 1000 with all elements set to `vdacs(1) = 3`
type the next command, depress and hold the push button, then press return or enter from the keyboard; hold the button until `meu2y` finishes; after the motor reaches steady state, read the generator voltage on the DMM, and record this value on the Data Sheet; the value will fluctuate so read the value to two digits past the decimal point; the motor runs for a total of 10 s and becomes steady shortly after startup.	
`[vgen,junk] = meu2y (vdac,vdac,1000*tstep,0)`	output `vdac` to the motor simultaneously capturing the generator output in the `vgen` column vector

`vdac = vdacs(2) * ones(length(t),1)`
`[vgen,junk] = meu2y (vdac,vdac,1000*tstep,0)`
`vdac = vdacs(3) * ones(length(t),1)`
`[vgen,junk] = meu2y (vdac,vdac,1000*tstep,0)`
`vdac = vdacs(4) * ones(length(t),1)`
`[vgen,junk] = meu2y (vdac,vdac,1000*tstep,0)`
	repeat for remainder of the test voltages; for each case, record the reading from the DMM on the Data Sheet to two digits past the decimal point; calculate the gain K and record this on the data sheet; note that the gain varies with input voltage indicating that the system is slightly nonlinear

6.3 Step Response

6.3.1 Obtain Response to a 4-V Step Input

(a) Use similar steps as above to write some MATLAB code to apply a 4-V step input to the motor and record and display the results in a plot. Plot both the input step and the tachometer output verses time on one plot. At the same time use the oscilloscope to display both the step and tachometer signals.

6.3.2 Time Constant from 63.2-% Point on Response Curve

(b) Use the X and Y cursors on the oscilloscope to (i) measure the steady-state gain and (ii) determine the time at which the change in the output voltage reaches 63.2 % of its final value. Here, time is measured with respect to the point at which the step change in the input voltage occurs.

Detailed Information for Measuring Time Constant Using Oscilloscope Cursors

(i) Turn the oscilloscope cursors on, and *select Channel 2 for the cursor source*. Align the Y cursor with the steady-state output voltage. Record this value as $V_{out}(\infty)$ on the Data Sheet. Use this value and the input step size of $V_{in}(t \geq 0) = 4$ V to compute $K_{63.2\%} = V_{out}(\infty) / V_{in}(t \geq 0)$.

(ii) Calculate $\Delta V_{out}(\tau) = 63.2\% \times V_{out}(\infty)$. Adjust the Y cursor to read this value.

(iii) Align the X1 cursor with the step change in the input voltage.

(iv) Align the X2 cursor with the intersection of the Y1(2) cursor and the output voltage. Record the Δt (2) cursor reading on the data sheet as $\tau_{63.2\%}$.

Important Note

The above method employs a single point on the step-response curve and thus exhibits maximum sensitivity to experimental and modeling errors. The value of 63.2 % is a convenient choice because (i) the time constant can be read directly and (ii) the slope of the step-response curve at this point facilitates accurate positioning of both cursors. What is most important, however, is that the point is representative of the step response. Because the step response for this case is $V_{out}(t) = V_{out}(\infty)[1 - \exp(-t/\tau)]$, the time constant is given by

$$\tau \quad = \quad -t / \ln[1 - V_{out}(t) / V_{out}(\infty)],$$

where t and $V_{out}(t)$ correspond to any point on the step-response curve. This method is described in greater detail in the Appendix C.

6.3.3 Time Constant from Intersection of Steady-state Asymptote and Tangent at t = 0

(c) The second method of determining the time constant τ is based on the slope of the tangent at t = 0. To execute this method, use the plot you created in MATLAB. You may have to zoom in on the initial slope to fit the line. We then print out this plot and draw (i) a tangent to the step response curve at t = 0 and (ii) a horizontal line at the steady-state voltage (steady-state asymptote) as shown below. The elapsed time between the step input and the point at which these two lines intersect is the time constant τ. The theory underlying this method is presented in the Appendix C.
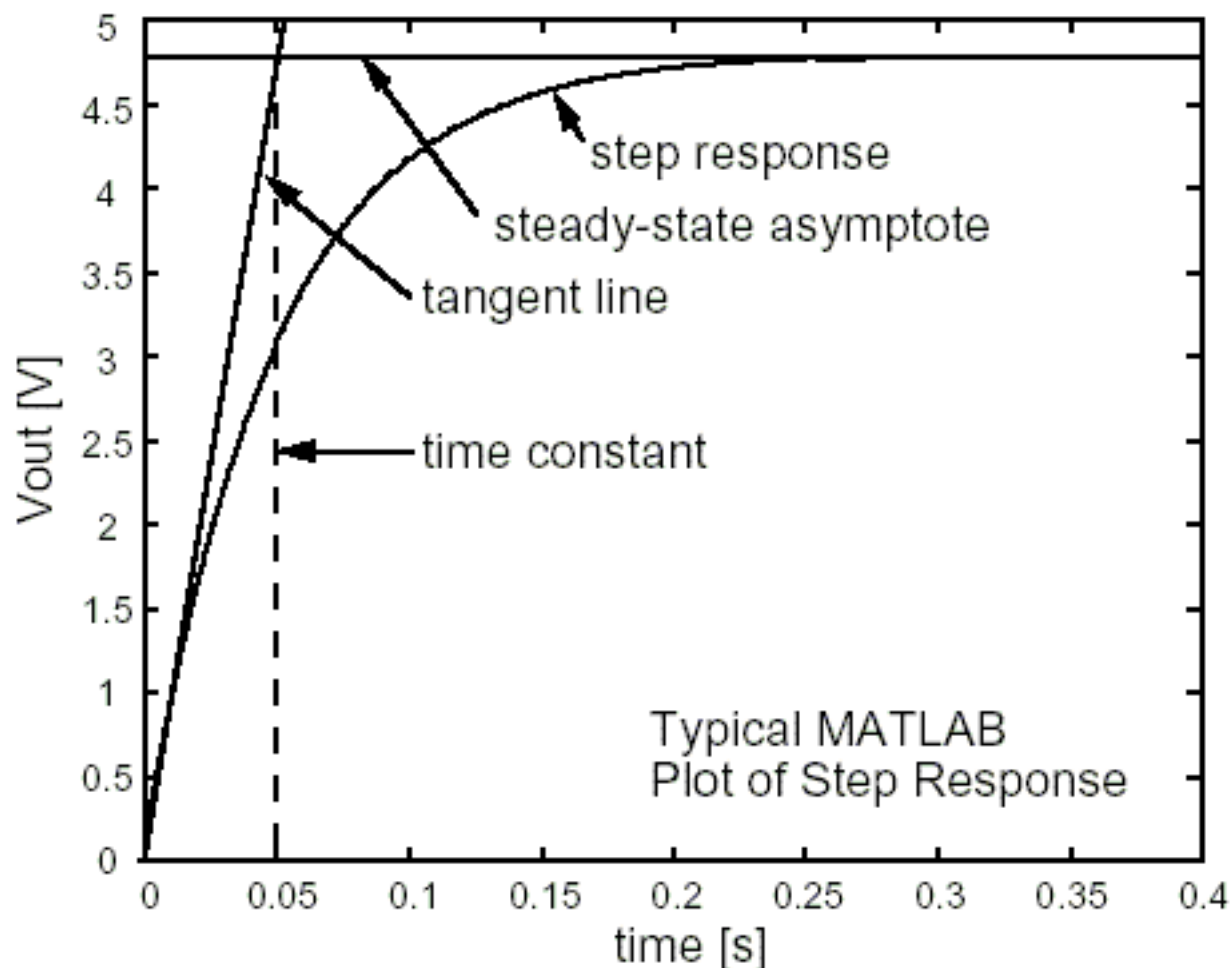
(d)   Perform the graphical analysis, and record this value of the time constant on the Data Sheet as $\tau_{tangent}$.

6.3.4 Time Constant from Integral of Step Response

(e)   The MATLAB commands below implement a method suitable for computer data analysis. This method uses all the data and is thus global. The method is based on integrating the step response with the `trapz` MATLAB function.

    (i)   Determine the final value of the step response (or steady-state output). Do this by averaging the last half of the collected data. Ask your TA to explain the ":" operator in MATLAB so you can access a subset of the collected data.

    (ii) Compute the steady-state gain based on the ratio of the steady-state output voltage to the step input voltage. Record this value on the Data Sheet as K_{int}.

    (iii) Use the MATLAB `help` capability to learn about the `trapz` function.

    (iv) Compute the system time constant using the integral method describe in Appendix C. Record this value on the Data Sheet as τ_{int}

6.3.5 Time Constant from a "Fit" of Step Response Data

(f)   The last step-response method involves plotting the observed response together with the predicted response based on K and τ found with other methods, then iterating on K and τ until good agreement between predicted and observed behavior is achieved.

Start So, start out with a K and τ from one of the above methods and in MATLAB calculate the predicted response (use the function "exp" for the exponential). On the same plot, plot the sampled data response and the predicted response verses time. Repeat with different values of K and τ until good agreement is achieved
commands,

Hint: Typing the first letter of the command followed by the up-arrow key returns to the last command starting with that letter.

Record the final values of K and τ on the Data Sheet as K_{fit} and τ_{fit}.

APPENDIX A. MOTOR-GENERATOR SYSTEM

A.1 Basic Layout

The system analyzed in this experiment consists of a DC motor coupled to a DC generator (a motor run backwards) as shown in Fig. A.1. The motor converts a voltage into angular speed, and the generator, in turn, converts this angular speed back into a voltage. The generator acts as a tachometer, a device for measuring rotational speed. Together, the motor-generator combination gives us a system that takes an input voltage and produces an output voltage. The rotational speed of the shaft serves as an intermediate variable of important physical significance. This voltage-in / voltage-out configuration is ideally suited to the DAC / ADC computer-based instrument available at the laboratory station.
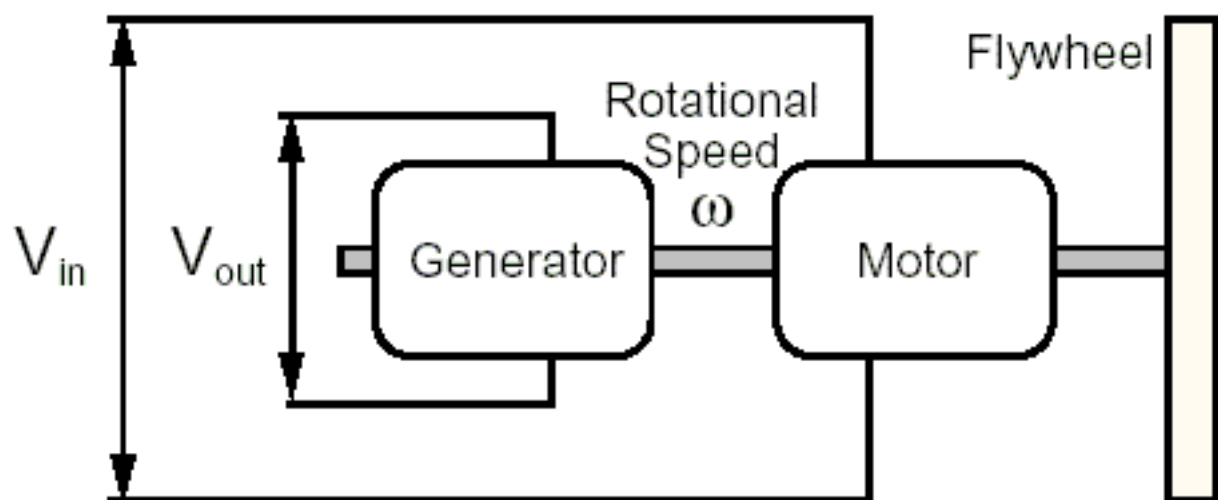

Figure A.1   Motor-generator system.

The motor and generator are connected to the computer as shown in Fig. A.2. Here, we see that the DAC output voltage is routed to a power amplifier rather than directly to the motor. The power amplifier is needed because the current-producing capability of the DAC is insufficient to drive the motor. Although the amplifier does provide a voltage gain of about 2.4, its primary purpose is to meet the power demands of the motor. The need for an intermediate component to handle the power requirements of the system arises frequently in real-world applications and illustrates an important principle in interfacing low-power electronic equipment with high-power mechanical devices.
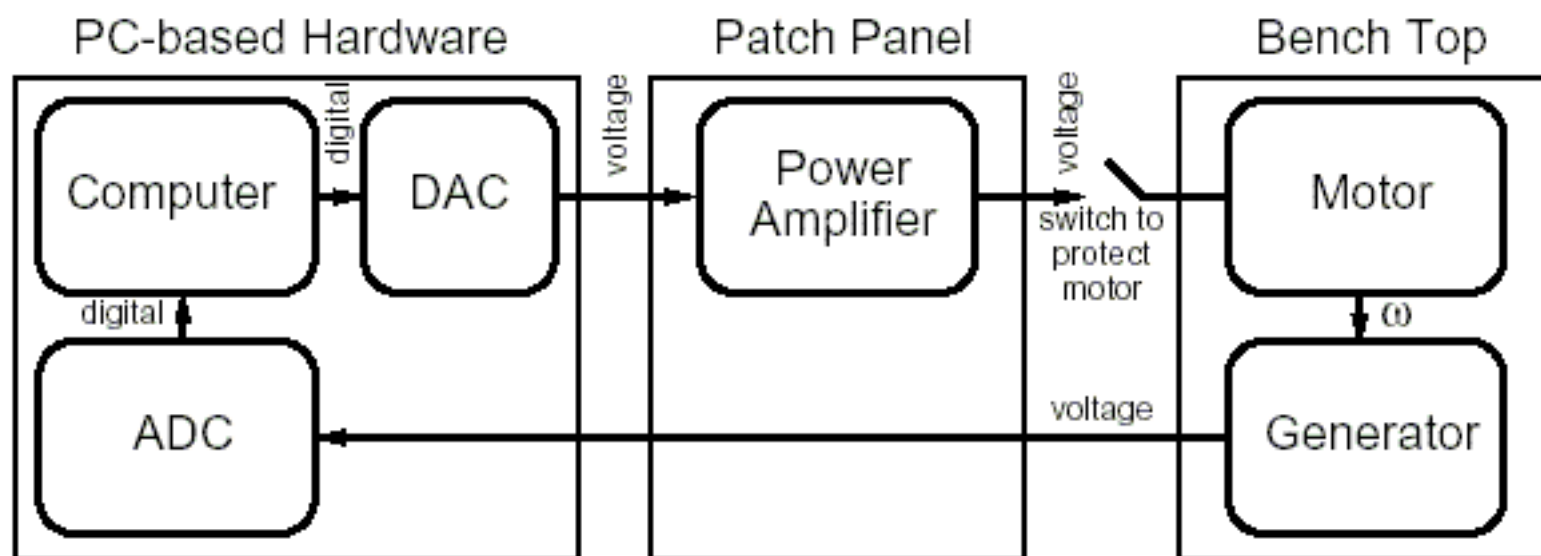

Figure A.2   Block diagram showing connections between instruments and motor-generator.

A.2 Dynamic Characteristics

Although the system used in this experiment consists of a motor and a generator connected together such that a voltage input produces a voltage output, our main interest lies in the way the motor speed responds to changes in input. We can ignore the dynamics characteristics of the generator and instead think of it simply as a speed sensor. To better understand the relationship between input and speed, consider the following simple

analogy from everyday life. Imagine driving on a level highway (I-57 or I-74, for example). The system here is the vehicle (analogous to the motor), the input is throttle position (analogous to input voltage), and the output is vehicle speed (analogous to motor speed or generator output). This is a first-order system with a characteristic time constant τ. We might be tempted to think that this is a second-order system because the equations of motion involve the second-order derivative acceleration. However, the system is actually first-order because acceleration is the *first-order* derivative of the system state variable speed. For the system to be *linear*, the force opposing the motion must be linearly proportional to vehicle speed. In the vehicle example, the resistive force is actually a constant (rolling friction) plus a second constant times the square of vehicle speed (aerodynamic drag) with no linear term whatsoever. Despite this inherently nonlinear behavior, dynamic response is very similar to that of a linear system. Indeed, because very few real-world systems are truly linear, the analysis considered in this experiment would be quite limited if the concepts could not also be extended to quasi-linear systems such as the vehicle example given here.

Vehicle speed does not respond instantaneously to changes in throttle position owing to vehicle inertia. Rather, as a first-order system, speed exhibits an exponential response to a step input and an exponential approach to a constant offset for a ramp input. This behavior is illustrated in Fig. A.3. We expect the same type of response for the motor used in this experiment.
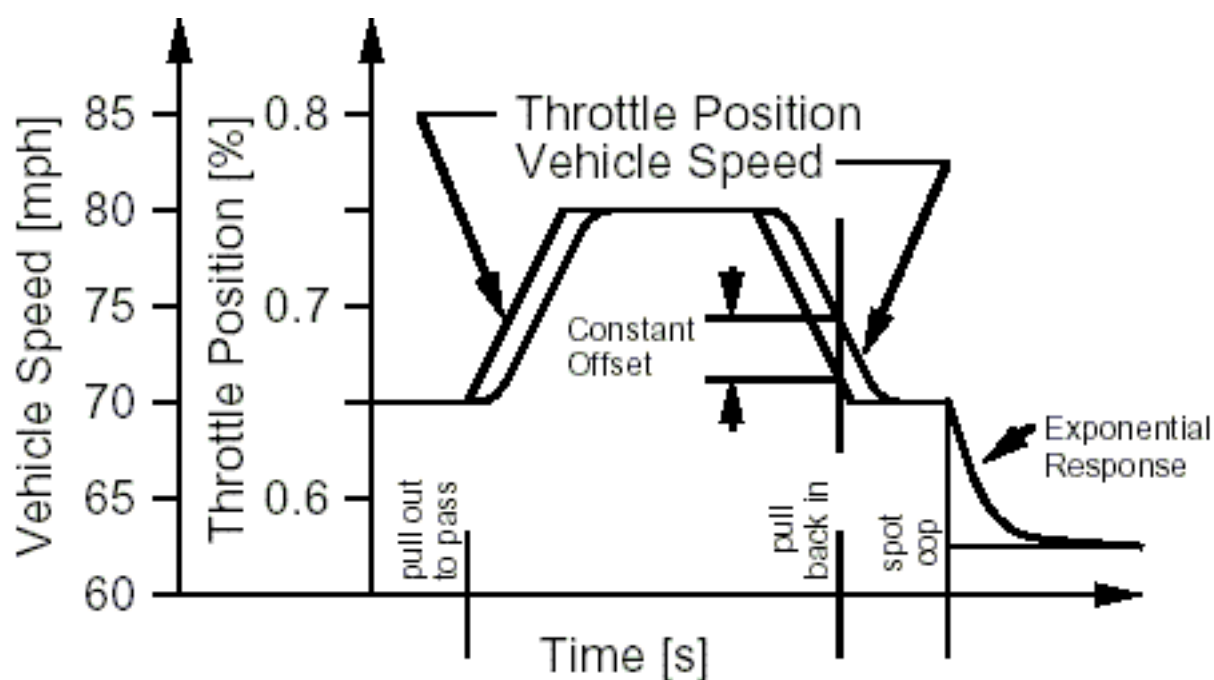

Figure A.3 Time history of throttle position and vehicle speed.

Returning to the motor-generator combination, the system is characterized by two parameters. One parameter is K, the steady-state gain or proportionality between input voltage and output voltage. Suppose that 1 V is applied to the input of the motor. After a while, the motor attains a certain constant speed, say 1200 rpm. The generator, in turn, converts this speed to an output voltage, say 1.2 V. The steady-state gain is then V_{out}/V_{in} = 1.2. If the system is linear, then an input voltage of 2 V to the motor produces an output voltage of 1.2 $\times$ 2 V = 2.4 V from the generator. Likewise, an input voltage of 0.5 V produces an output voltage of 1.2 $\times$ 0.5 V = 0.6 V. For a linear system, we have that V_{out}/V_{in} = K = 1.2 for all V_{in}. As we shall see later, K varies slightly with input voltage. The second parameter is the time constant τ that characterizes the dynamic response of the system. The time constant depends on the system inertia or "mass" and the relationship between energy losses (i. e., conversion to thermal energy) and angular velocity. Here, the system "mass" is actually the moment of inertia of the rotating components, and the losses arise from both electrical and mechanical "friction" effects.

The purpose of this experiment is to measure K and τ from the dynamic response of the motor-generator system using the two methods previously described; namely, step response and sinusoidal frequency response. Before presenting these methods in detail, let us examine the system more closely to better understand its dynamic behavior.

A.3 Block Diagram, Governing Equations, and Transfer Function

Figure A.4 is an expanded block diagram that sets forth the various elements of the motor so that its overall dynamic behavior can be more easily analyzed. The computer generates a digital representation of the input waveform that is sent to the digital-to-analog converter (DAC). The DAC converts this digital form to an analog

voltage V_{DAC} between -10 and +10 V. The power amplifier increases this voltage by the amplifier gain K_{amp} that is nominally 2.4. This voltage is, in turn, applied to input terminals of the motor. To understand how this applied voltage creates the current necessary to drive the motor, we must consider the electric circuit of the motor as shown in Fig. A.5. This simple circuit consists of three components: (a) a resistor representing the resistance of the motor coil, (b) an inductor representing the inductance of the motor coil, and (c) a voltage source representing the reverse voltage produced by the motor. The reverse voltage is proportional to the speed at which the shaft turns and, in effect, represents the generator effect of the motor. Using Kirchoff's Voltage Law along with the component equations for a resistor and inductor, we have the governing equation for the electrical circuit of the motor. Thus,
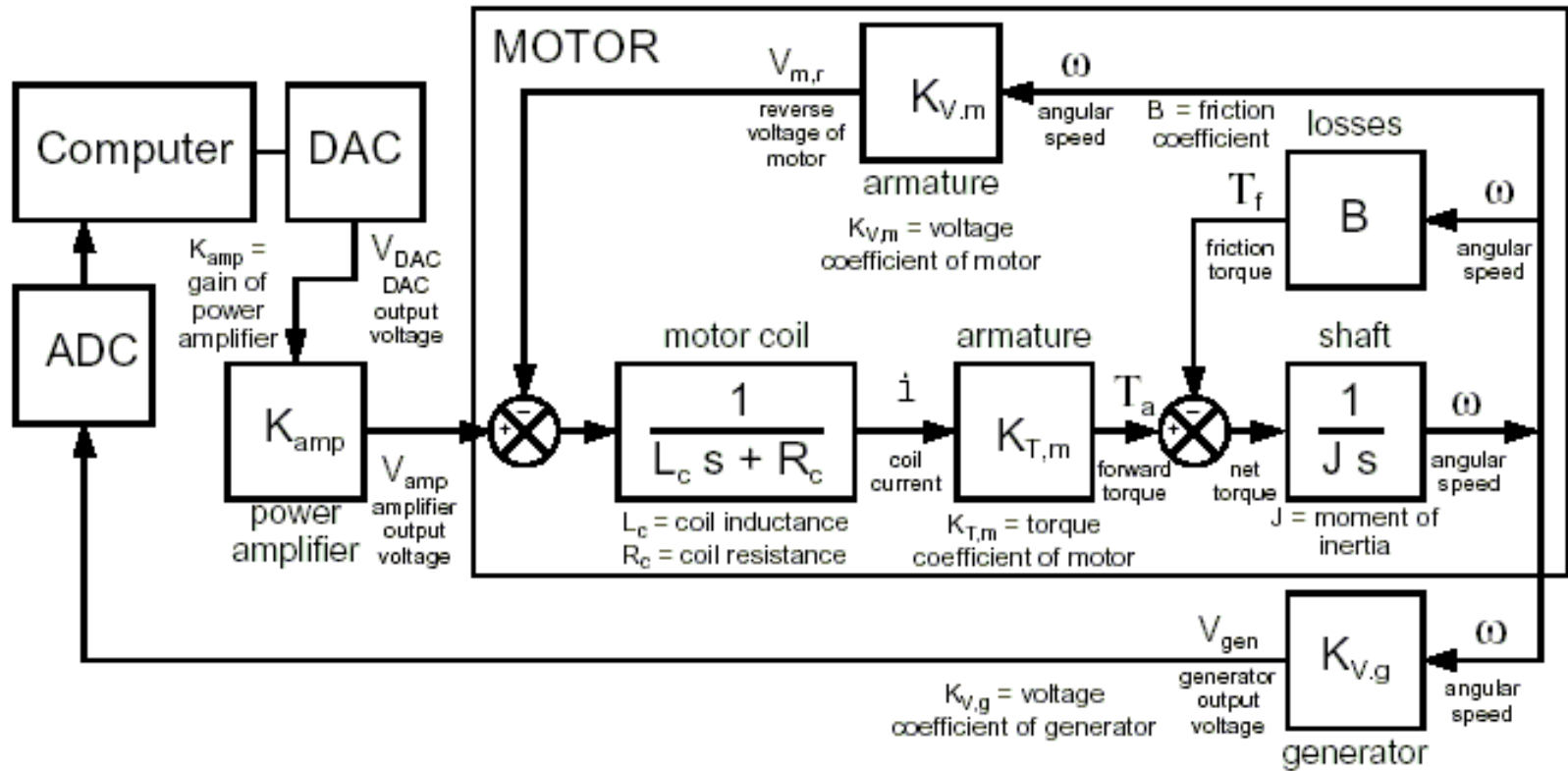

Figure A.4   Block diagram of motor-generator system showing main elements of dynamic response.
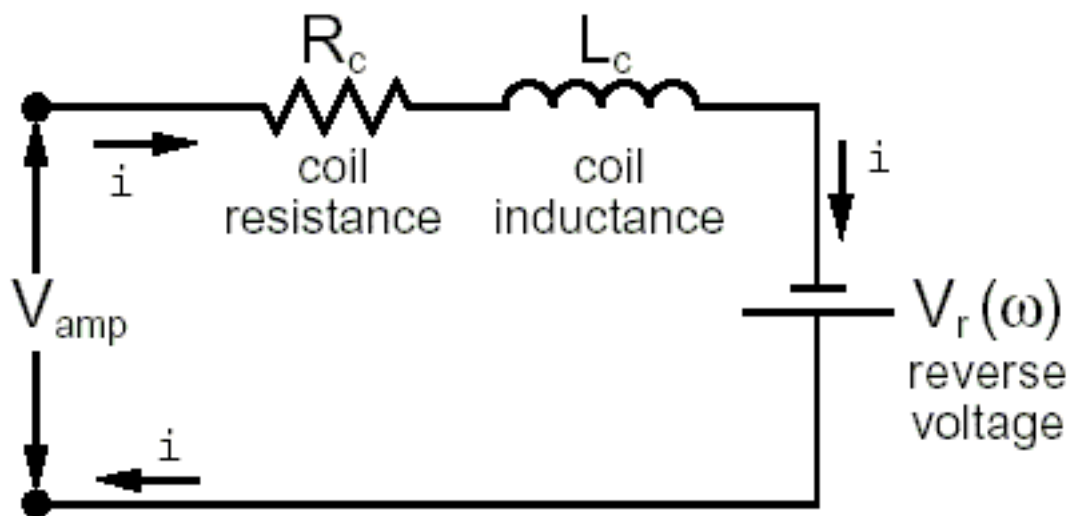
Figure A.5   Electric circuit of motor.

Governing Equation for Electrical Circuit of Motor

$$i(t)\, R_c \;+\; L_c\, \frac{di(t)}{dt} \;=\; V_{amp}(t) \;-\; V_r(t)$$

We can solve this equation in the time domain using any of several methods or in the transform domain using either (a) the Laplace transform Method for an initial-value problem or (b) the Fourier Transform Method for a quasi-steady periodic input. The advantage of the transform domain is that each equation is converted to an

algebraic form that can, in turn, be written as a transfer function. The various transfer functions representing the governing equations for the different parts of the overall system can then be combined using block algebra.

The corresponding Laplace and Fourier Transforms of the above governing equation are:

Laplace Transform of Governing Equation for Electrical Circuit of Motor

$$L_c \, s \, \hat{i}(s) \; - \; i(0) \; + \; R_c \, \hat{i}(s) \; = \; \hat{V}_{amp}(s) \; - \; \hat{V}_r(s)$$

Fourier Transform of Governing Equation for Electrical Circuit of Motor

$$L_c \, (j \, \omega) \, \tilde{i}(\omega) \; + \; R_c \, \tilde{i}(\omega) \; = \; \tilde{V}_{amp}(\omega) \; - \; \tilde{V}_r(\omega)$$

Solving for $\hat{i}(s)$ and $\tilde{i}(\omega)$, we have

Laplace Transfer Function

$$\hat{i}(s) \; = \; \frac{V_{amp}(s) - V_r(s)}{L_c \, s \, + \, R_c} \; + \; \frac{i(0)}{L_c \, s + R_c} \qquad\qquad \hat{G}(s) \; = \; \frac{1}{L_c \, s \, + \, R_c}$$

Fourier Transfer Function

$$\hat{i}(\omega) \; = \; \frac{\tilde{V}_{amp}(\omega) - \tilde{V}_r(\omega)}{L_c \, (j\omega) \, + \, R_c} \qquad\qquad \tilde{G}(s) \; = \; \frac{1}{L_c \, (j\omega) \, + \, R_c}$$

We observe that the two transfer functions are the same except that (a) $(j\omega)$ is substituted for s and (b) the initial condition term is absent from the Fourier Transfer form. Following the standard convention in linear systems analysis, we shall write the transfer function in Laplace form but without the initial-value term. This may seem a bit inconsistent at first, but we must remember that the purpose of the block diagram is to allow us to easily visualize the relationships between the various elements of system. Cluttering up the diagram with subtle distinctions and messy extra terms only serves to obscure the relationships and thus defeat the basic purpose of the block diagram. We must always keep these simplifications in the back of our mind, however.

We shall further simplify the notation by using the same variable for both its time-dependent function and its Laplace or Fourier transform. Thus, "i" may represent $i(t)$, $\hat{i}(s)$, or $\tilde{i}(\omega)$ depending on context. Again, we choose simplicity of notation over precise definitions to improve the overall clarity of the expression.

Simplified Form of Transfer Function of Motor Coil

$$i \; = \; \frac{V_{amp} - V_r}{L_c \, s \, + \, R_c}$$

Consider again the system block diagram shown in Fig. A.4. We next consider the following three simple proportional relationships.

Current-to-Torque Conversion

$$T_a \; = \; K_{T,m} \, i$$

where T is armature torque, i is coil current, and $K_{T,m}$ is the torque coefficient of the motor. $K_{T,m}$ is taken to be a constant of the motor.

Reverse Voltage of Motor

$$V_r \; = \; K_{V,m} \, \omega$$

where V_r is reverse voltage of the motor, ω is the angular speed of the motor, and $K_{V,m}$ is the voltage coefficient of the motor. $K_{V,m}$ is also taken to be a constant of the motor.

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Reverse Voltage of Generator

$$V_{gen} = K_{V,g}\, \omega$$

where V_{gen} is reverse voltage of the generator, ω is the angular speed of the generator, and $K_{V,g}$ is the voltage coefficient of the generator. Note that the generator follows the same analysis as the motor. However, because the generator feeds the high-impedance inputs of the ADC, oscilloscope, and multimeter, the current is nearly always zero, and the output voltage is just the reverse voltage of the generator.

The remaining two boxes relate to the mechanical portion of the system; namely, (a) the rotating shaft, (b) the shaft bearings and other elements that provide the frictional load, and (c) the flywheel.

Governing Equation for Rotating Shaft

Applying Newton's Law of Motion to a rotating body, we have

$$J\, \frac{d\omega(t)}{dt} = T_a(t) - T_f(t)$$

where $\omega(t)$ is angular speed, $T_a(t)$ is armature torque, $T_f(t)$ is frictional torque, and J is the moment of inertia of the shaft and flywheel. The corresponding simplified transfer function is

$$\omega = \frac{T_a - T_f}{Js}\, .$$

Frictional Torque

Lastly, the frictional torque is proportional to angular speed as follows.

$$T_f = B\, \omega$$

where T_f is frictional torque, ω is the angular speed of the generator, and B is the coefficient of the kinetic friction. B is taken to be a constant representing the total bearing friction in the motor, generator, and flywheel.

Now we can combine all the various elements to obtain a single transfer function between V_{DAC} and V_{gen}. We begin with the mechanical elements of the system and work backward to V_{DAC} and forward to V_{gen}. Thus,

$$\frac{V_{gen}}{V_{DAC}} = \frac{K_{amp}\, K_{V,g}\, K_{T,m}}{(L_c\, s + R_c)(J\, s + B) + K_{V,m}\, K_{T,m}} = \frac{K}{(\tau_1 s + 1)(\tau_1 s + 1)}$$

where

$$K = \frac{K_{amp}\, K_{V,g}\, K_{T,m}}{K_{V,m}\, K_{T,m} + R_c\, B}\, ,$$

$$\tau_1 = \tau_m \left(\frac{2\, \rho_e}{1 + \rho_e - \sqrt{(1 + \rho_e)^2 - 4\, \rho_e\, K_1}} \right)\, ,$$

$$\tau_2 = \tau_m \left(\frac{2\, \rho_e}{1 + \rho_e + \sqrt{(1 + \rho_e)^2 - 4\, \rho_e\, K_1}} \right)\, ,$$

$$K_1 = \frac{K_{V,m}\, K_{T,m} + R_c\, B}{R_c\, B}\, ,$$

$$\tau_m = \frac{J}{B}\, , \qquad \tau_e = \frac{L_c}{R_c}\, , \qquad \text{and} \qquad \rho_e = \frac{\tau_e}{\tau_m}\, .$$

Here, K is the steady-state gain, τ_m is the mechanical time constant of the system, and τ_e is the electrical time constant of the system. As shown, the system is second-order; however, because $\tau_e \ll \tau_m$, we have the limiting case of $\rho_e \to 0$ for which

$$\tau_1 \to \tau = \frac{\tau_m}{K_1} = \frac{R_c\, J}{K_{V,m}\, K_{T,m} + R_c\, B} \qquad \text{and} \qquad \tau_2 = 0\, .$$

The corresponding transfer function is

$$\frac{V_{gen}}{V_{DAC}} = \frac{K}{\tau\, s + 1}\,.$$

A listing of important system parameters is given in Table A.1. Here, we see that ρ_e is very small and that the assumption of first-order system behavior is thus very good.

Table A.1   Parameters of Motor-generator System.

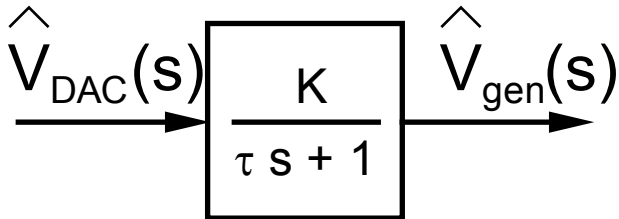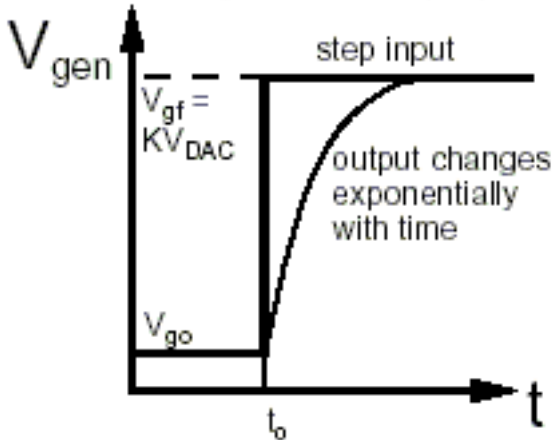| Symbol	Description	Value	Units
$K_{T,m}$	Torque coefficient of the motor [1]	0.0572	N-m/A
$K_{V,m}$	Voltage coefficient of motor [1]	0.0573	V-s
R_c	Coil resistance of motor [1]	3.76	Ω
L_c	Coil inductance of motor [1]	0.00125	H
B_m	Friction coefficient of motor [1]	2.43×10^{-5}	N-m-s
J_m	Moment of inertia of motor [1]	2.65×10^{-6}	kg-m^2
$K_{V,g}$	Voltage coefficient of generator [1]	0.0287	V-s
J_g	Moment of inertia of generator [2]	2.65×10^{-6}	kg-m^2
K_{amp}	Gain of power amplifier [3]	2.4	V/V
J_{fly}	Moment of inertia of fly wheel [4]	5.10×10^{-5}	kg-m^2
J_{coup}	Moment of inertia of couplers [4]	3.06×10^{-6}	kg-m^2
B_{other}	Friction coefficient of components other than motor [5]	1×10^{-5}	N-m-s
J	Moment of inertia of system = $J_g + J_{fly} + 3\,J_{con}$ [6]	6.55×10^{-5}	kg-m^2
B	Friction coefficient of system = $B_m + B_{other}$ [7]	3.43×10^{-5}	N-m-s
τ_e	Electrical time constant of system [7]	3.41×10^{-4}	s
τ_m	Mechanical time constant of system [7]	1.91	s
K_1	Time constant reduction factor [7]	27.05	–
ρ_e	Time constant ratio [7]	1.78×10^{-4}	–
K	Steady-state gain of system [7]	1.156	V/V
τ_1	Primary time constant of system [7]	70	ms
τ_2	Secondary time constant of system [7]	0.3	ms

Notes:

1. Based on manufacturer's specifications.
2. Estimated to be the same as the motor.
3. Parameter of power amplifier.
4. Calculated from physical dimensions and material properties of flywheel and connector.
5. Estimated. As B is increased, the (primary) system time constant decreases.
6. Other components make negligible contribution to moment of inertia.
7. Calculated from equations of system model.

APPENDIX B. FIRST-ORDER SYSTEM BEHAVIOR

The important dynamic characteristics of a first-order system are summarized in Table B.1.

Table B.1   Summary of First-order System Analysis.

Governing Equation	
Normalized Form	$\dfrac{dV_{gen}}{dt} = \dfrac{KV_{DAC}(t) - V_{gen}(t)}{\tau}$ $\quad V_{gen}(0) = V_{go} \quad \tau = $ time constant
Laplace Transform Form	$\hat{V}_{gen}(s) = \hat{G}(s)\left[V_{DAC}(s) + \dfrac{\tau V_o}{K}\right] \qquad \hat{G}(s) = \dfrac{K}{\tau\,s + 1}$
Fourier Transform Form	$\tilde{V}_{gen}(\omega) = \tilde{G}(\omega)\,\tilde{V}_{DAC}(\omega) \qquad \tilde{G}(\omega) = \dfrac{K}{\tau\,(j\,\omega) + 1}$
Transfer Function Form	$\hat{V}_{DAC}(s) \longrightarrow \boxed{\dfrac{K}{\tau s + 1}} \longrightarrow \hat{V}_{gen}(s)$

Solutions	
General Time-Domain Solution	$V_{gen}(t) = \exp(-t/\tau)\displaystyle\int_0^t \exp(t'/\tau)\,K\,V_{DAC}(t')\,dt' \;+\; V_{go}$

Steady-state Input	
Input $V_{DAC}(t)$	$V_{DAC}(t) = V_{DAC} = $ constant
Output $V_{gen}(t)$	$V_{gen}(t) = K\,V_{DAC}$

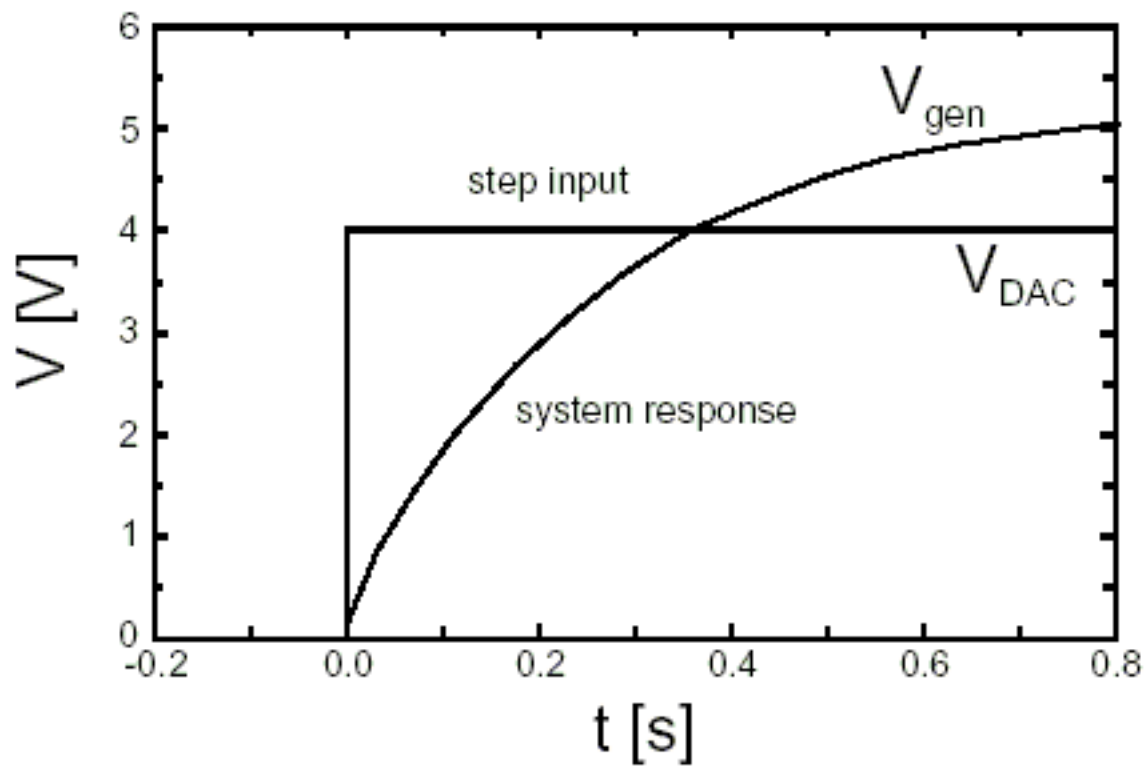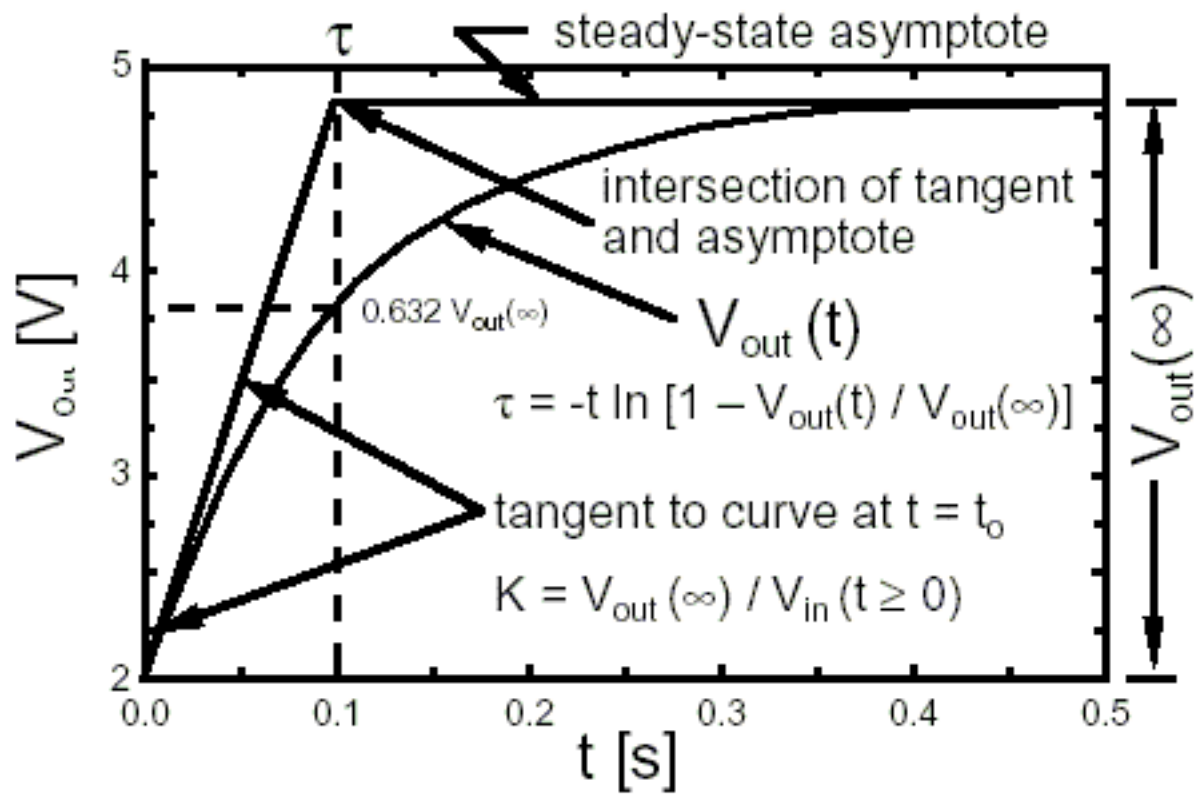| Step Input	
Input $V_{DAC}(t)$	$V_{DAC}(t) = V_{DAC} = $ constant $\quad t > t_o \quad V_{gen}(0) = V_{go}$
Output $V_{gen}(t)$	$V_{gen}(t) = \underbrace{K\,V_{DAC}}_{\text{steady state}} + \underbrace{(V_{go} - K\,V_{DAC})\exp\left[-(t-t_o)/\tau\right]}_{\text{exponential decay of initial offset}}$

Figure C.1   Typical step-input response.

Figure C.2   Two graphical methods of determining system time constant.

APPENDIX C. STEP RESPONSE

In our experiment, we make a step change in the input voltage and observe how the system responds. The expected waveform is shown in Fig. C.1. Here, we have an exponential approach to a new steady state speed and generator output voltage.

We use four algorithms to extract values for the steady-state gain K and time constant τ from the step response.

Step Response Method 1:   Time constant is time required for voltage to reach 63.2 % of its final value

This method is illustrated in Fig. C.2

The parametric step response of the output voltage for the special case of $V_o = 0$ is given by

$$V_{out}(t) \quad = \quad V_{out}(\infty) \, [\, 1 - \exp(-t / \tau)\,]$$

$$V_{out}(\tau) \quad = \quad V_{out}(\infty) \, [\, 1 - \exp(-1)\,] = \quad 0.632 \, V_{out}(\infty)$$

As previously noted, the 63.2-% point is a particularly convenient choice because (a) the time constant can be read directly and (b) the slope of the curve at this point facilitates graphical analysis. However, other representative points can be used. In this case, the time constant τ is calculated from

$$\tau \quad = \quad -t / \ln[\, 1 - V_{out}(t) / V_{out}(\infty)\,] \,.$$

Step Response Method 2:   Intersection of steady-state asymptote and tangent at t = 0

This method is also illustrated in Fig. C.2. A tangent is constructed at t = 0, and the time constant is the intersection of this line with the steady-state asymptote defined by $V = V_{out}(\infty)$. From the parametric step response, we determine that the slope of the tangent at t = 0 is given by

$$\left. \frac{dV_{out}}{dt} \right|_{t\,=\,0} \quad = \quad \frac{V_{out}(\infty)}{\tau} \quad .$$

The equation of the tangent line is then

$$V_{tangent}(t) \quad = \quad \frac{V_{out}(\infty)}{\tau} \, t \,.$$

The intersection of the tangent and the asymptote is then given by

$$V_{tangent}(t) \quad = \quad \frac{V_{out}(\infty)}{\tau} \, t \quad = \quad V_{out}(\infty) \qquad or \qquad t = \tau \,.$$

For both Method 1 and Method 2, K is just the ratio of the output voltage to the input voltage at steady state. Thus,

$$K \quad = \quad \frac{V_{out}(\infty)}{V_{in}(t \geq 0)}$$

Step Response Method 3:   Integral of step response

A decided advantage of the graphical algorithms used in Methods 1 and 2 above stems from the fact that a person can quickly recognize and potentially compensate for a host of real-world problems. A "good" tangent or a "good" point can be selected even if the data are noisy. Reducing this level of sophistication to a numerical algorithm that the computer can execute is a challenge indeed. On the other hand, numerical methods are desirable because graphical methods are inherently subjective.

As we shall see, the output of the motor-generator system is quite noisy even when the input is clean. This fact is not accounted for in our simple linear model of the system. The model handles a noisy input but does not account for the noise produced by the motor itself. We need a numerical method that minimizes the effect of this noise. Because integrals tend to reduce noise, Method 3 is based on the integrated signal.

From the parametric step response

$$V_{out}(t) = V_{out}(\infty) [1 - \exp(-t/\tau)]$$

we can readily show that

$$\tau = \int_0^{\infty} \left[1 - \frac{V_{out}(t)}{V_{out}(\infty)}\right] dt \quad .$$

The integral can be computed numerically from the response data.

Step Response Method 4:   Iterative fit of response data

In this method, we plot the parametric response curve together with the response data using guessed values of K and τ and iterate on the parameters until good agreement is achieved. This method is easier to implement than a conventional curve fit and furthermore demonstrates the effect that K and τ each have on the system response.

Appendix D.   MATLAB Functions

In this experiment, we use the MATLAB software to generate the desired input waveforms and control the PC-based hardware. We employ several M-files specifically designed to ease this process. A brief discussion of these M-files and the associated MATLAB functions is in order.

D.1 M-file to Control the PC-based Hardware

The operation of the analog-to-digital and digital-to-analog converters is controlled by the following MATLAB function.

```
[y0, y1] = meu2y (u0, u1, stepms, echo)
```

"`meu2y`" simultaneously (a) samples Analog Input Channel 0 storing the results in column vector "`y0`", (b) samples Analog Input Channel 1 storing the results in column vector "`y1`", (c) outputs the column vector of voltages "`u0`" to Analog Output Channel 0, and (d) outputs the column vector of voltages "`u1`" to Analog Output Channel 1. "`stepms`" is the time step in ms, and "`echo`" is a flag that allows the input to be echoed to the output. We normally do not use the echoing capability of the "`meu2y`" function, and thus this argument is set to 0. Sampling begins a short delay time after the "`meu2y`" command is entered. The "`meu2y`" function executes as many time steps as there are elements in the "`u0`" and "`u1`" column vectors.

D.2 Standard MATLAB Functions

We also use the following standard MATLAB functions.

mean(V)   computes the mean of array of values V

trapz(t, V)   computes an approximation of the integral of V with respect to t using the Trapezoidal Rule; t and V are vectors of the same length containing discrete values of the integration variables

To learn more about these built-in functions, type "`help mean`" or "`help trapz`" at the MATLAB prompt.