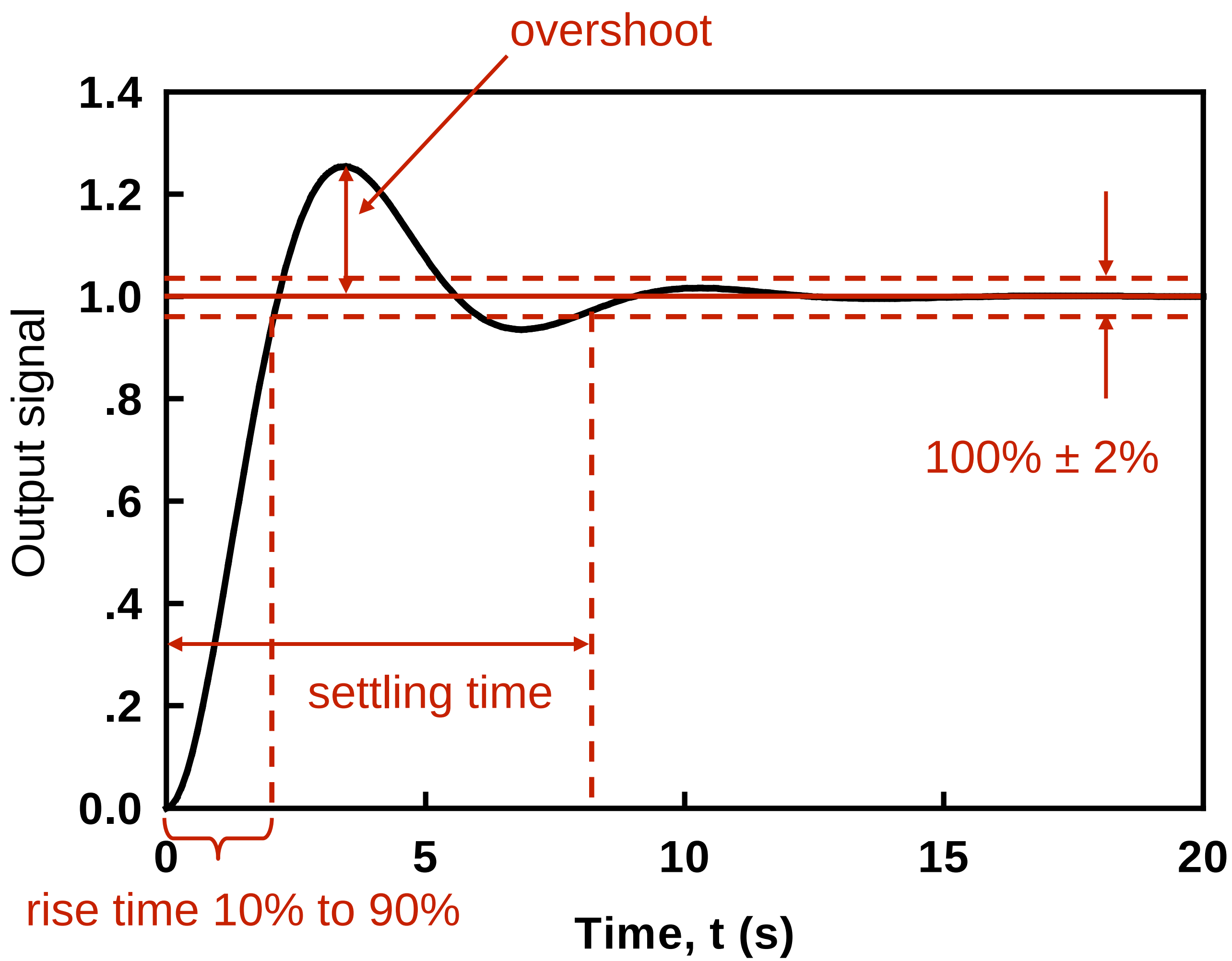
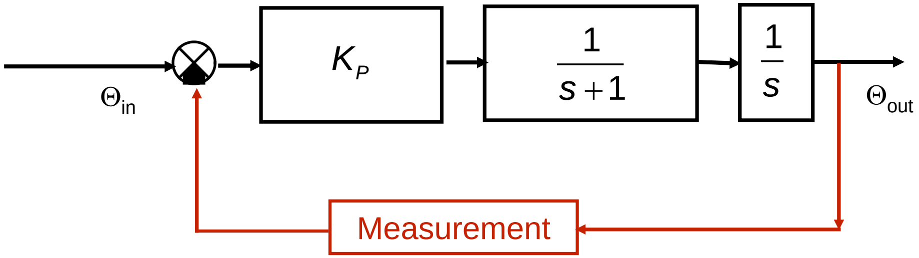
Second-Order Step Response



Matlab & Controller Design



Plant: $G(s) = \dfrac{1}{(s+1)s}$

Compensator: $D(s) = K_P$

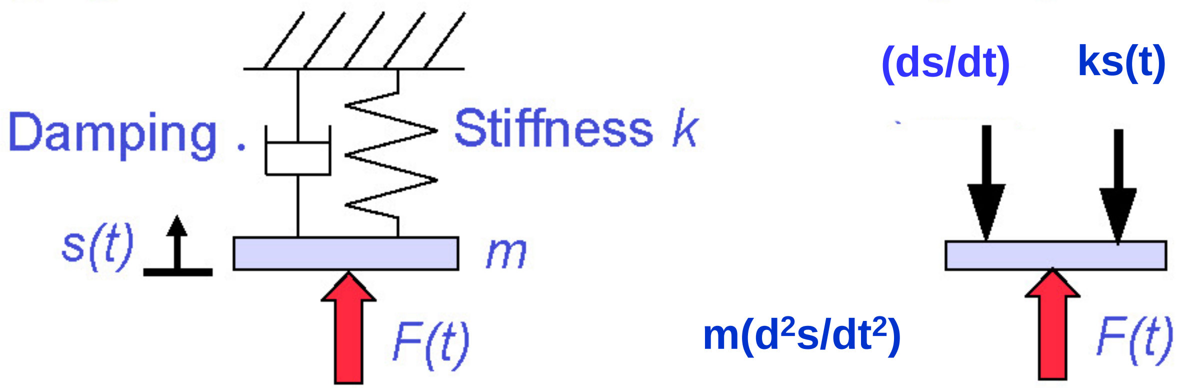
Transfer Function: $$TF = \frac{K_P\left(\dfrac{1}{(s+1)s}\right)}{1+K_P\left(\dfrac{1}{(s+1)s}\right)}$$

Matlab:

```
sysGs = tf([1], [1 1 0]);
rltool(sysGs);
```


Second-Order Systems

- Recall: A simple mechanical system:

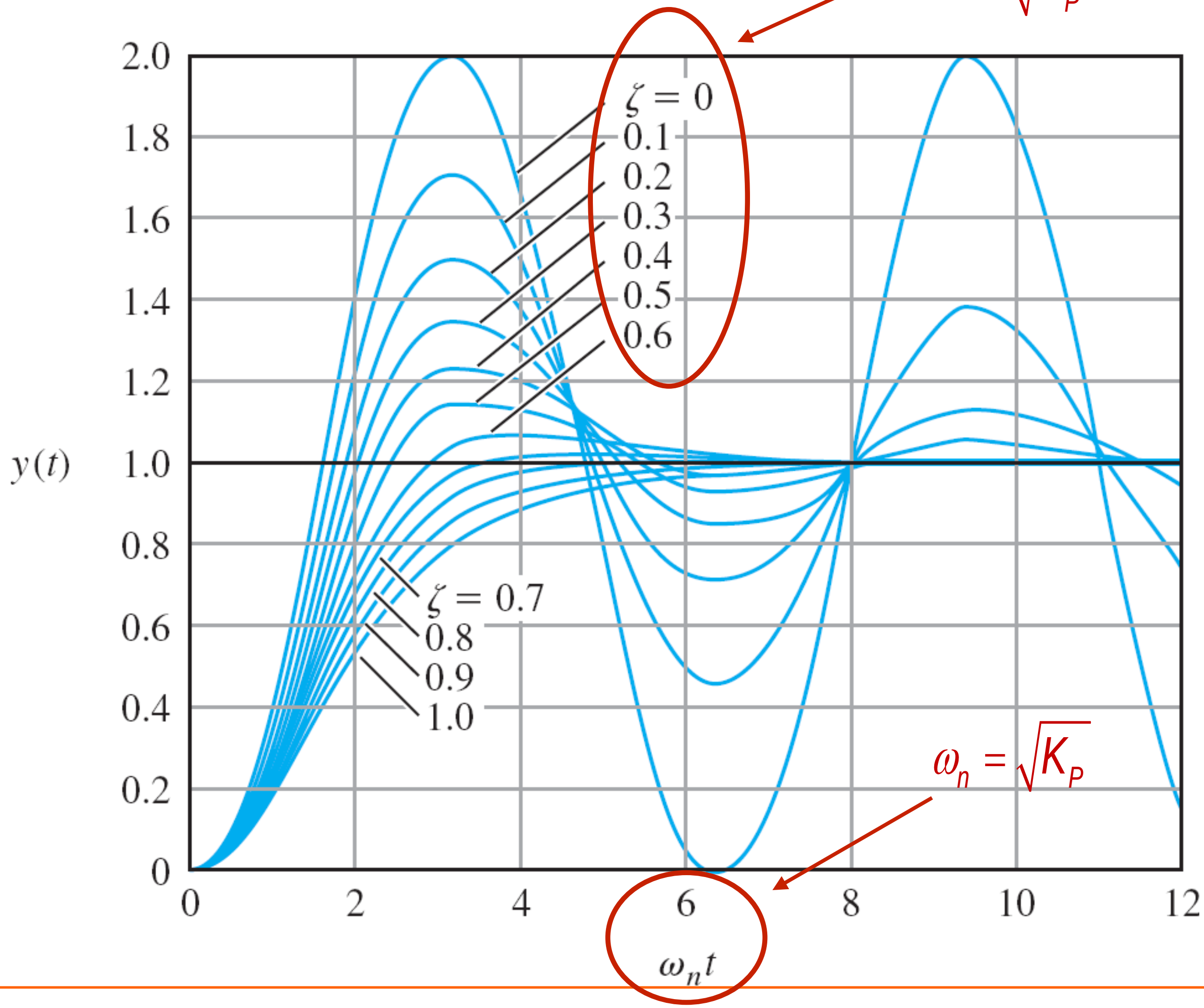


$$m\frac{d^2s}{dt^2}+c\frac{ds}{dt}+ks=F(t)$$

$K=\dfrac{1}{k}$ m/N $\qquad \zeta=\dfrac{c}{2\sqrt{km}}$ (dimensionless) $\qquad \omega_n=\sqrt{\dfrac{k}{m}}$ rads/s

Effect of (s) Pole Positions

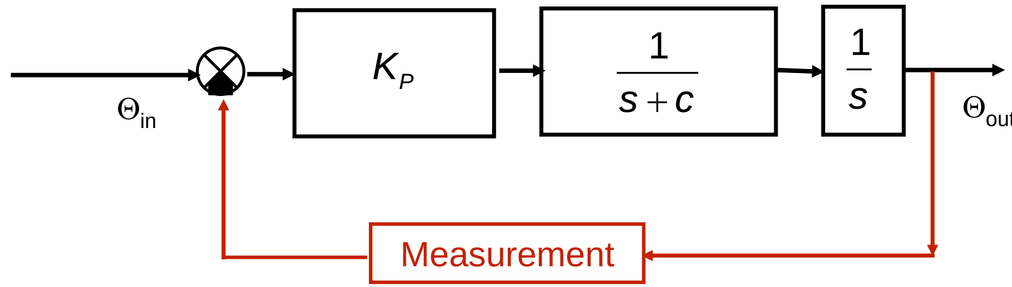
$\zeta = \frac{c}{2\sqrt{K_P}}$



$\omega_n = \sqrt{K_P}$

Effect of (s) Pole Positions

- System Block Diagram



- Closed-Loop Transfer Function:

$$TF = \frac{K_P}{s^2 + sc + K_P}$$

- Corresponding Differential Equation:

$$\frac{d^2\theta_{out}}{dt^2} + c\frac{d\theta_{out}}{dt} + K_P\theta_{out} = K_P\theta_{in}$$

$$K = 1 \qquad \zeta = \frac{c}{2\sqrt{K_P}} \qquad \omega_n = \sqrt{K_P}$$

Express the TF denominator as:

$$\left(s + \sigma - j\omega_d\right)\left(s + \sigma + j\omega_d\right)$$

$$= s^2 + 2\sigma s + \left(\sigma^2 + \omega_d^2\right)$$

$$\sigma = \zeta\omega_n$$

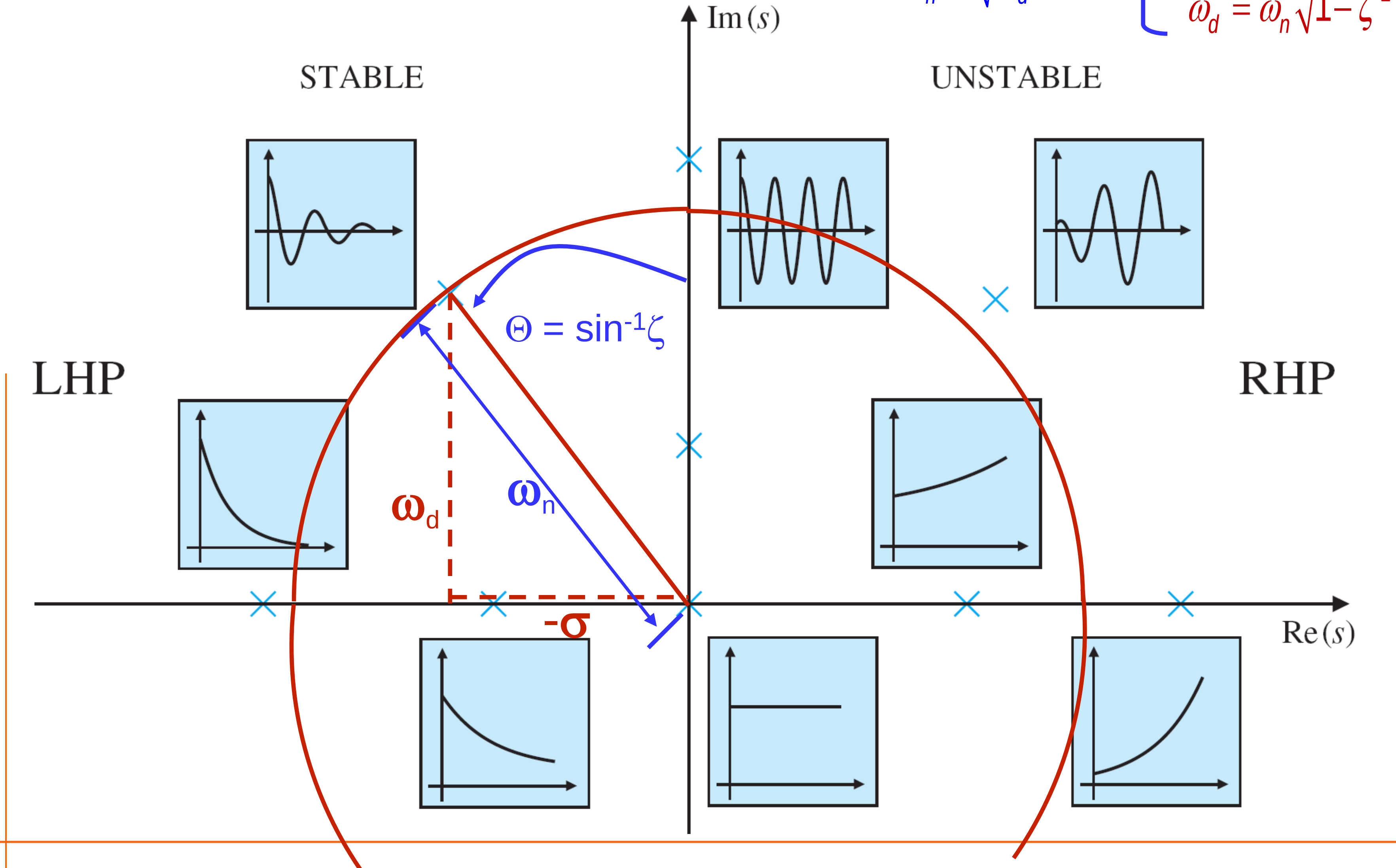
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$s = -\sigma + j\omega_d$

Effect of (s) Pole Positions

$\omega_n = \sqrt{\omega_d^2 + \sigma^2}$

$$\begin{cases} \sigma = \zeta\omega_n \\ \omega_d = \omega_n\sqrt{1-\zeta^2} \end{cases}$$



Z-Transform

- Discrete analog to the continuous Laplace Transform
- Transforms linear difference equations to algebraic equations that are much easier to solve
- Enables dynamic analysis of discrete systems using familiar transform tools (root locus, Nyquist, Bode)

$$Z\left\{f\left(k\right)\right\}=F\left(z\right)=\sum_{k=0}^{\infty}f\left(k\right)z^{-k}$$

- This leads directly to:

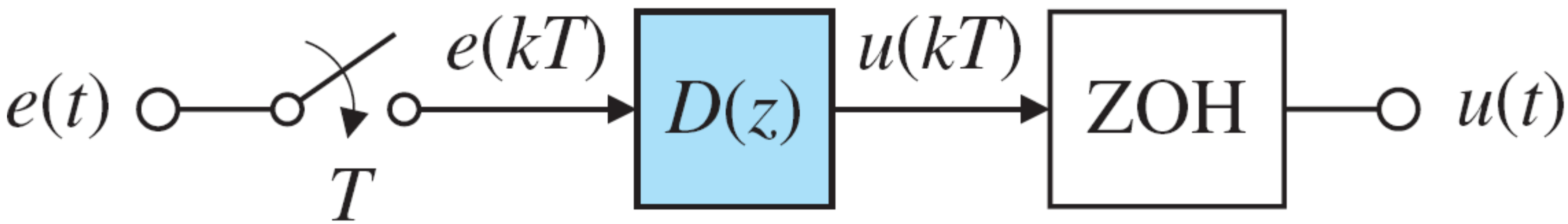
$$Z\left\{f\left(k-1\right)\right\}=z^{-1}F\left(z\right)$$

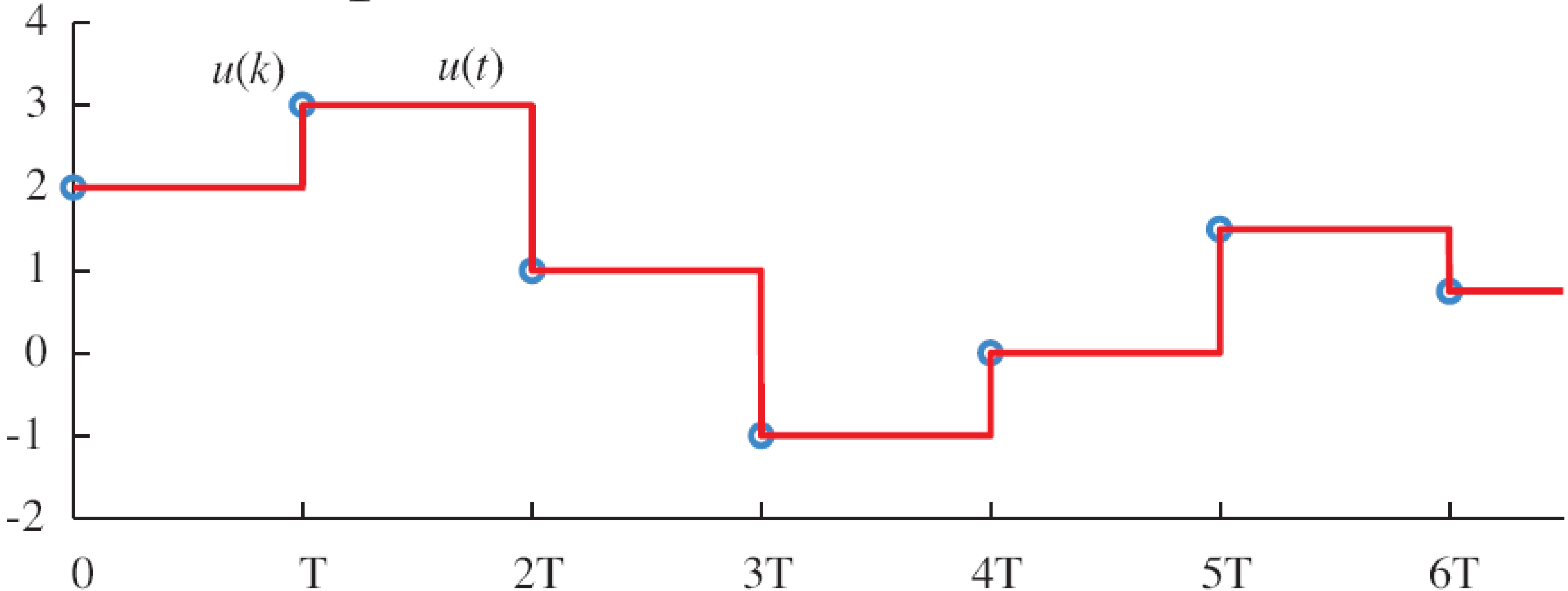
z^{-1} can be considered a shift operator

This is analogous to the Laplace transform as a derivative operator: $\mathcal{L}\left\{\dot{f}(t)\right\}=sF\left(s\right)$

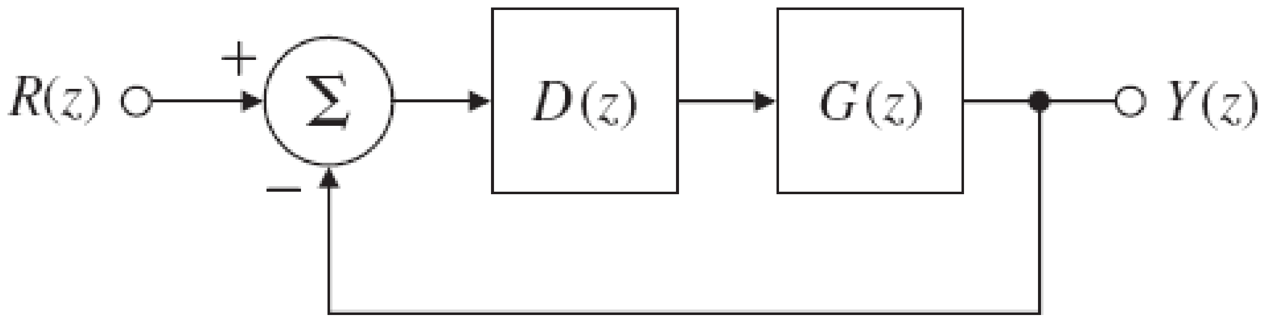
Zero-Order Hold

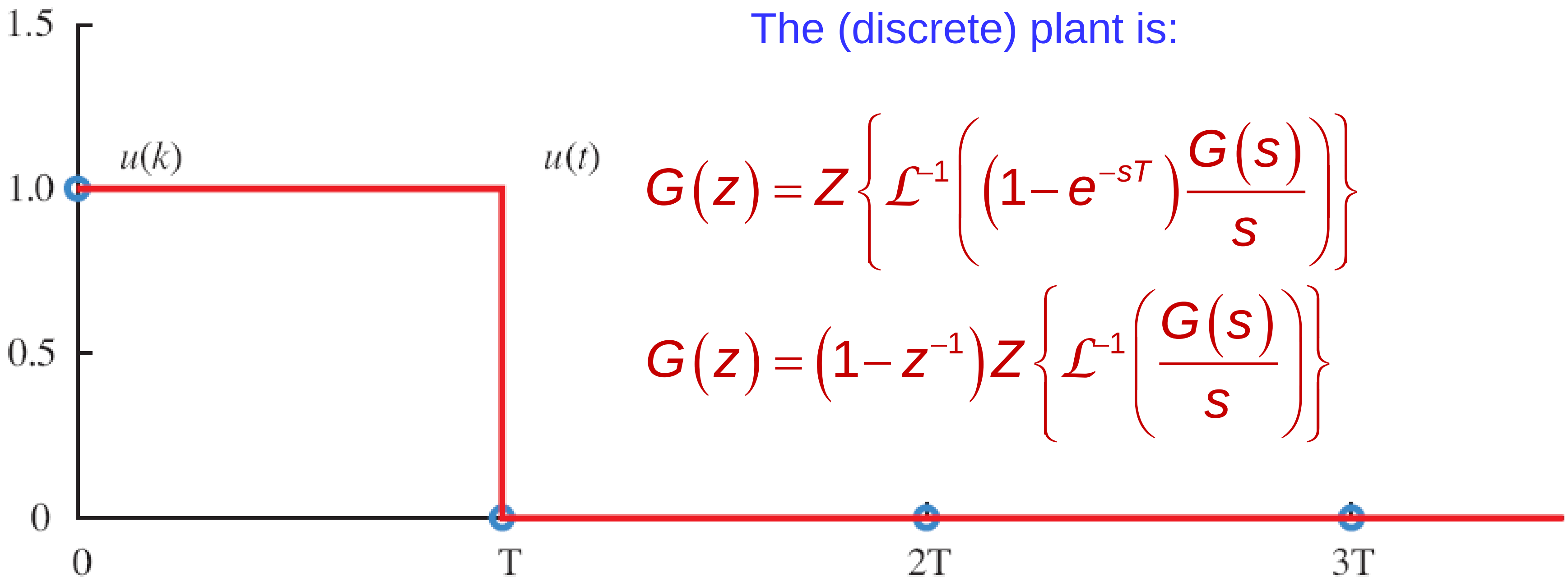
- A simple model of a typical digital to analog converter:





Zero-Order Hold





The (discrete) plant is:

$$G(z) = Z\left\{\mathcal{L}^{-1}\left(\left(1-e^{-sT}\right)\frac{G(s)}{s}\right)\right\}$$

$$G(z) = \left(1-z^{-1}\right)Z\left\{\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)\right\}$$

Relation between z and s

$$Z\left[f\left(kt\right)\right]=\sum_{k=0}^{\infty}e^{-akT}z^{-k}$$
$$=1+\left(\frac{e^{-aT}}{z}\right)^{1}+\left(\frac{e^{-aT}}{z}\right)^{2}+\ldots$$
$$=\frac{1}{1-\left(\frac{e^{-aT}}{z}\right)}$$

- Consider the continuous signal:

$$f\left(t\right)=e^{-at},t>0$$

- The corresponding Laplace transform is:

$$F\left(s\right)=\frac{1}{s+a}$$

A pole at s = -a in the s-plane corresponds to a pole at $z = e^{-aT}$ in the z-plane

- The corresponding z-transform is:

$$F\left(z\right)=\frac{z}{z-e^{-aT}}$$

- In general, characteristics in the z-plane are related to those in the s-plane by:

$$z=e^{sT}$$

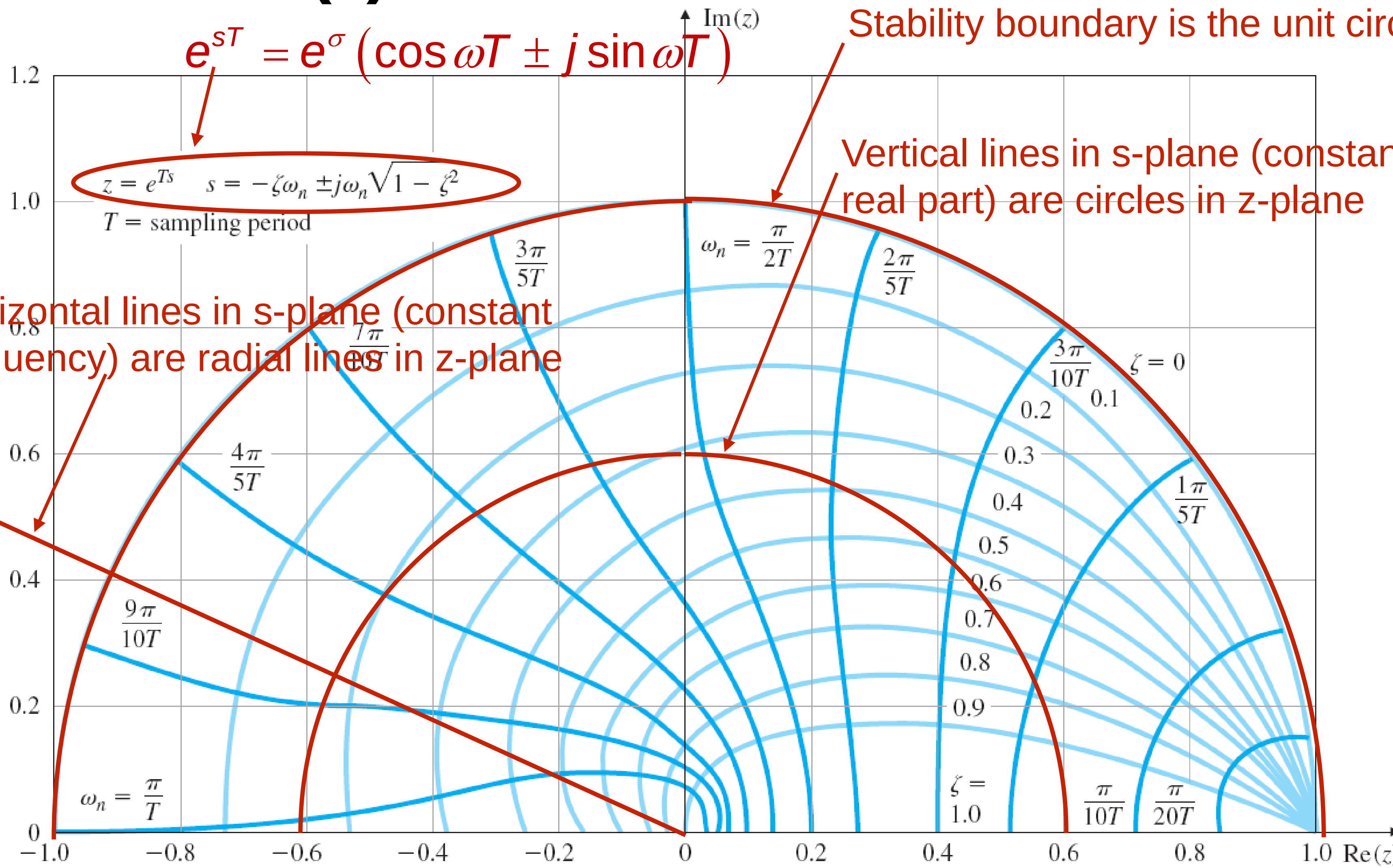
Effect of (z) Pole Positions

$$e^{sT} = e^{\sigma}\left(\cos\omega T \pm j\sin\omega T\right)$$

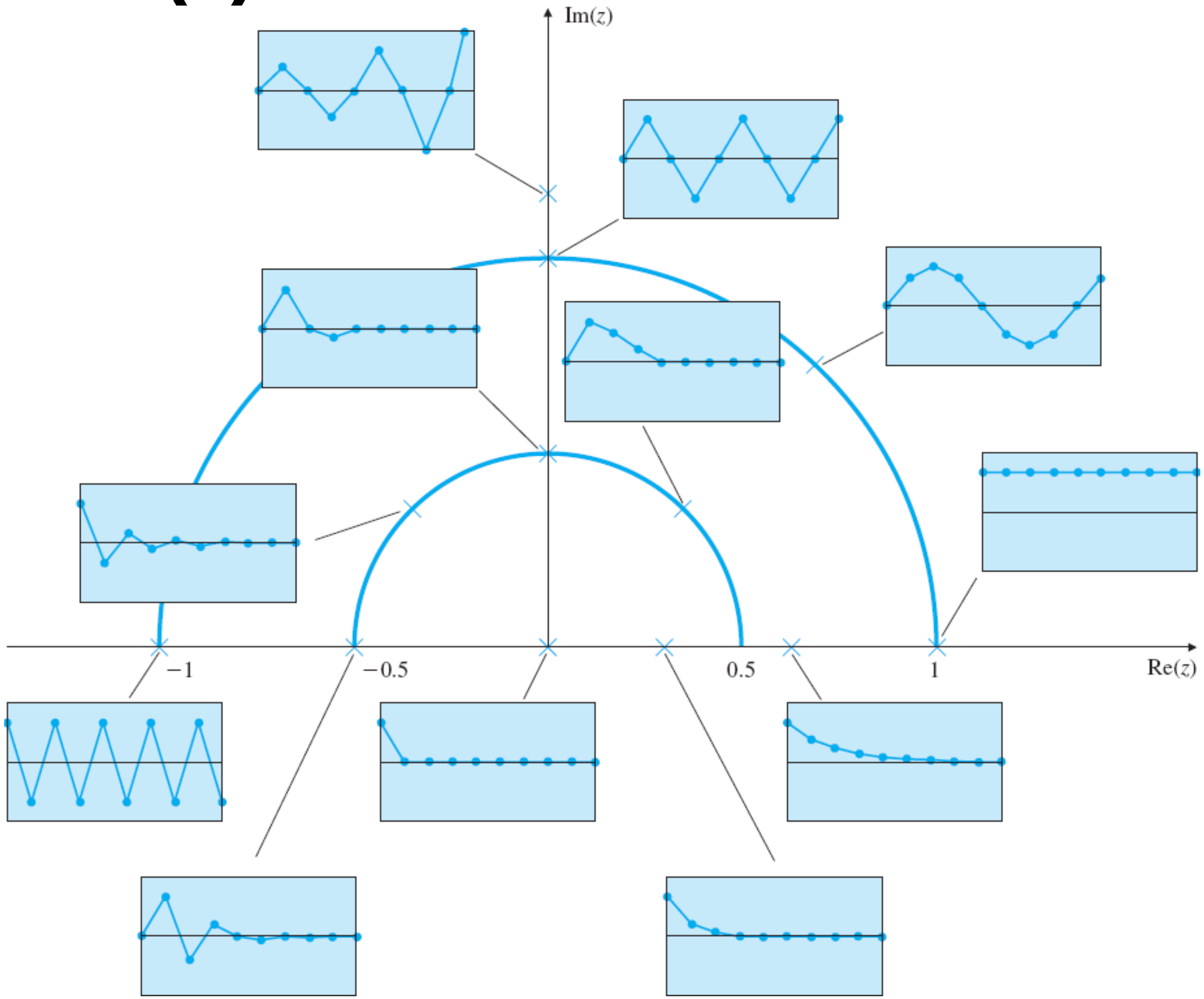
Stability boundary is the unit circle

Vertical lines in s-plane (constant real part) are circles in z-plane

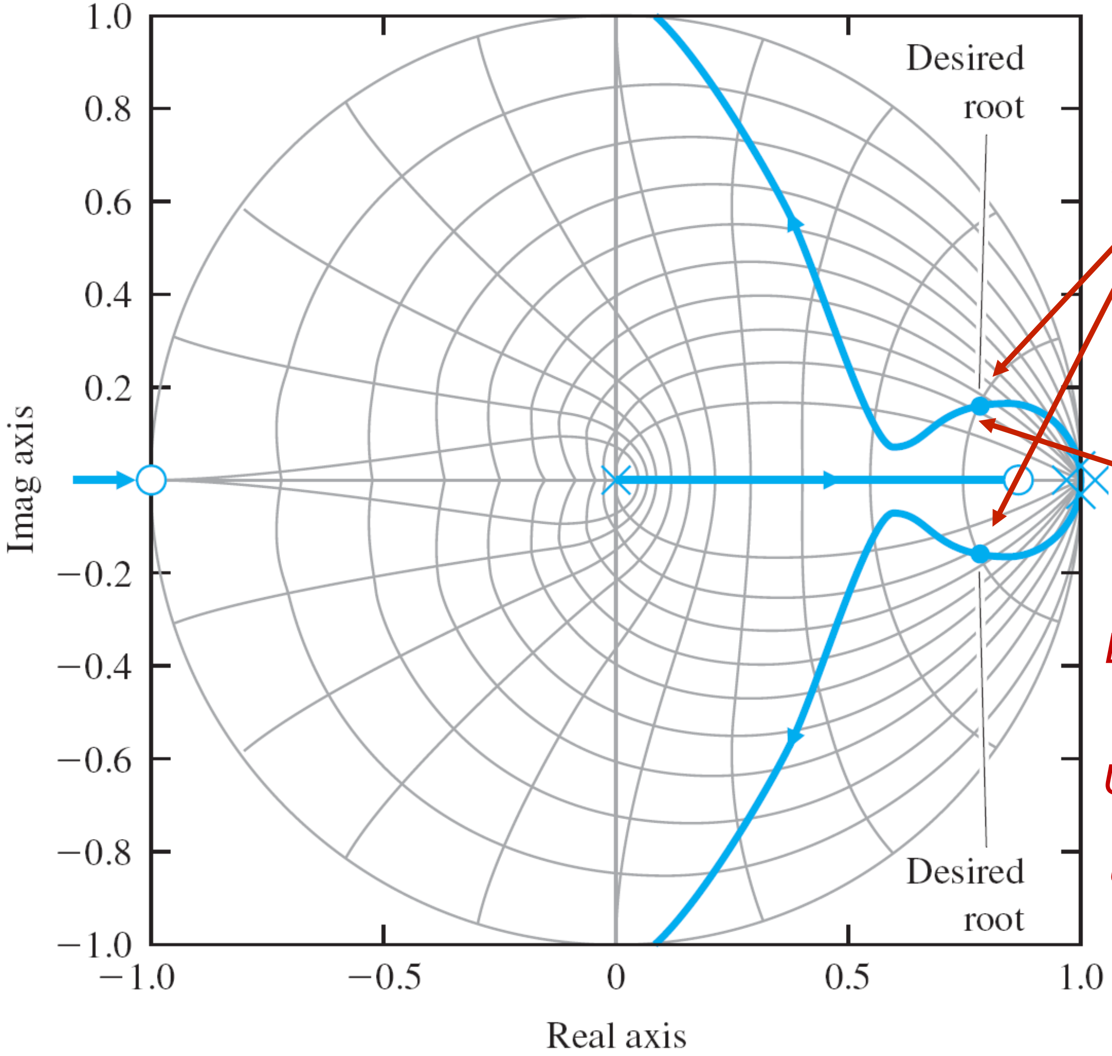
Horizontal lines in s-plane (constant frequency) are radial lines in z-plane



Effect of (z) Pole Positions



Discrete Analysis



$z = 0.78 \pm j0.18$

$K_1 = 0.374$

$$D(z) = 0.374\frac{z-0.85}{z}$$

$$U(z) = 0.374\left(1-0.85z^{-1}\right)E(z)$$

$$u(k) = 0.374e(k) - 0.318e(k-1)$$