Observers and Observer-Based Controller Design in Discrete Space
Taken from a document written by William Perkins

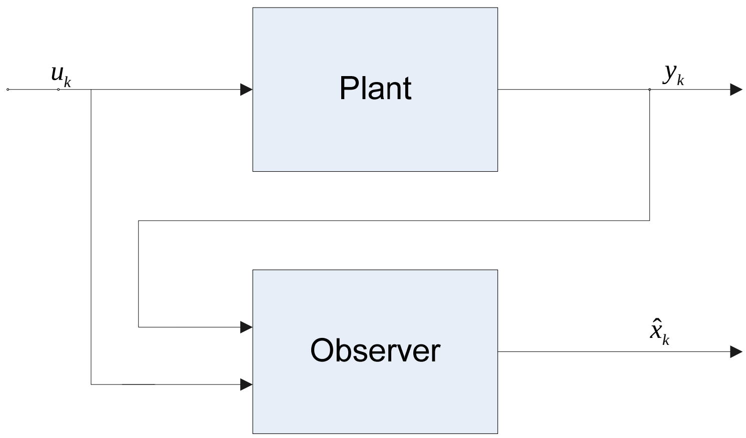
These notes will walk you through a control design for a plant with the discrete state space model

$$x_{k+1} = A_d x_k + B_d u_k; \qquad y_k = Cx_k; \qquad x_k(0) = x_0$$

A. Observer

Problem: A_d, B_d, C known, *y* and *u* measured (observed), *x0* unknown. Find estimate $\hat{x}_k(t)$ of $x_k(t)$ such that $x_k(t) - \hat{x}_k(t) \to 0$ "quickly" at $t \to \infty$.

Solution: Full-order Observer:



Try the following dynamic system as an observer.

$$\hat{x}_{k+1} = A_d \hat{x}_k + B_d u_k + L(y_k - \hat{y}_k),$$
$$\hat{y}_k = C\hat{x}_k,$$

where L is an observer gain matrix, to be chosen by the designer.

Does it work? Let $\tilde{x}_k = x_k - \hat{x}_k$. Then

$$\tilde{x}_{k+1} = x_{k+1} - \hat{x}_{k+1}$$
$$\tilde{x}_{k+1} = A_d x_k + B_d u_k - A_d \hat{x}_k - B_d u_k - L(Cx_k - C\hat{x}_k)$$
$$\tilde{x}_{k+1} = A_d(x_k - \hat{x}_k) - LC(x_k - \hat{x}_k) = (A_d - LC)\tilde{x}_k$$

So $\tilde{x}_k = 0$ is an equilibrium state of the observer. The dynamic response of observers depends on the eigenvalues of $(A_d - LC)$. The eigenvalues of $(A_d - LC)$ are the same as those of

$$(A_d - LC)^T = (A_d^T - C^T L^T).$$

Eigenvalues can be place arbitrarily if and only if (A_d^T, C^T) is *controllable*:

$$S = [C^T \mid A_d^T C^T \mid \ldots \mid (A_d^T)^{n-1} C^T].$$

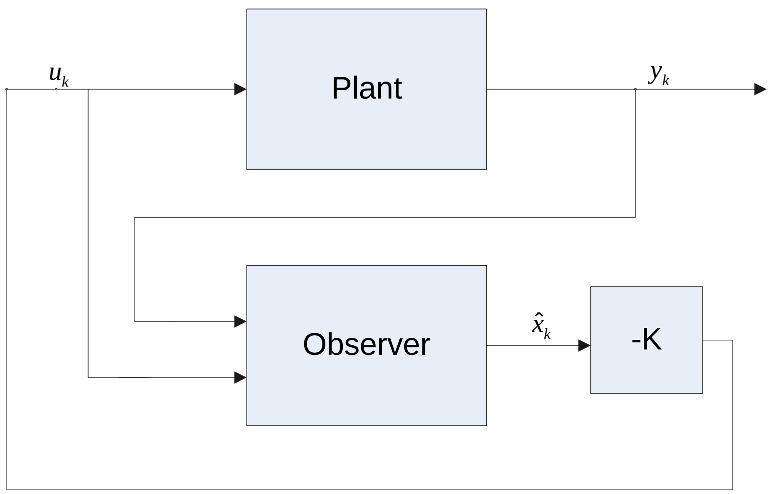
The rank of S is equal to the rank of its transpose, which is denoted

$$O = \begin{bmatrix} C \\ CA_d \\ \vdots \\ CA_d^{n-1} \end{bmatrix}.$$

Thus the observer eigenvalues can be placed arbitrarily by selection of the observer gain matrix L if and only if the rank of O is n. In this case the state space model is called *observable*.

B. Controller Design: Combined Observer and Observer Feedback

$$u_k = -K\hat{x}_k$$



The observer error is

$$\tilde{x}_{k+1} = (A_d - LC)\tilde{x}_k$$

And hence the control looks like a perturbation of full state feedback:

$$u_k = -K\hat{x}_k = -Kx_k + K\tilde{x}_k$$

$$\begin{bmatrix} x_{k+1} \\ \tilde{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A_d - B_d K & B_d K \\ 0 & A_d - LC \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix} = A_{cl} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix}$$

The eigenvalues of A_{cl} equal the eigenvalues of $(A_d - B_d K)$ plus the eigenvalues of $(A_d - LC)$. ***Separation!***

For implementation (or simulation), use x_k and $\hat{x}_k$:

$$\begin{aligned} x_{k+1} &= A_d x_k + B_d u_k, \qquad y_k = Cx_k \\ \hat{x}_{k+1} &= A_d \hat{x}_k + B_d u_k + L(y_k - C\hat{x}_k) \\ u &= -K\hat{x}_k \end{aligned}$$