SWINGING UP A PENDULUM BY ENERGY CONTROL[*]

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Abstract: This paper presents the concept of energy control and shows how robust strategies for swinging up an inverted pendulum are obtained using this idea. The behavior obtained with the strategy depends critically on the ratio of the maximum acceleration of the pivot to the acceleration of gravity. A comparison with with minimum time strategies gives interesting insights into the robustness issues.

Keywords: Inverted pendulum, swing-up, energy control, minimum time control.

1. INTRODUCTION

Swinging up an inverted pendulum is a classic experiment that is used in control laboratories, see Furuta et al. (1992). It can be accomplished by minimum time control. This strategy is, however, not very robust. In this paper the problem is approached as a problem of energy control. The pendulum is simply controlled in such a way that its energy is driven towards a value equal to the steady-state upright position. The pendulum then approaches the upright position where it can be caught with an appropriate strategy. The analysis shows that the problem is completely characterized by two parameters. One is a time scale and the other is the ratio of the maximum acceleration of the pivot and the acceleration of gravity. The results can be generalized to multiple pendulums.

2. PRELIMINARIES

Consider a single pendulum. Let its mass be m and let the moment of inertia with respect to the pivot point be J. Furthermore let l be the distance from the pivot to the center of mass. The angle between the vertical and the pendulum is θ, where θ is positive in the

clock-wise direction. The acceleration of gravity is g and the acceleration of the pivot is u. The equations of motion of the pendulum is

$$J_p \ddot{\theta} - mgl \sin \theta + mul \cos \theta = 0 \qquad (1)$$

The system has two state variables, the angle θ and the rate of change of the angle $\dot{\theta}$. It is natural to let the state space be a cylinder. In this state space the system has two equilibrium points corresponding to $\theta = 0$, $\dot{\theta} = 0$, and $\theta = \pi$, $\dot{\theta} = 0$. The model given by Equation (1) is based on several assumptions, friction has been neglected and it has been assumed that the pendulum is a rigid body. It has also been assumed that there is no limitation on the velocity of the pivot.

Normalization

The model given by Equation (1) has five parameters, the moment of inertia J_p, the mass m, the length l, the acceleration of gravity g and the maximum acceleration of the pivot, which is denoted as ng. Normalized variables are useful to characterize the properties of a system. The parameter n is dimension free. Furthermore we introduce $\omega_0 = \sqrt{mgl/J_p}$, which is the frequency of small oscillations around the downward position. The equations of motion are then characterized by two parameters only.

It is natural to choose the energy so that it is zero in the upright position and to normalize it by mgl, which is the energy required to raise the pendulum from the downward to the horizontal position. The normalized energy can then be written as

$$E = mgl \left(\frac{1}{2} \left(\frac{\dot{\theta}}{\omega_0} \right)^2 + \cos \theta - 1 \right) \qquad (2)$$

Controllability

Local controllability means that the linearized model is controllable. The model given by Equation (1) is locally controllable when $\theta \neq \pi/2$, i.e., except when the pendulum is horizontal. Since the control signal is bounded to ng, we also find that if the angular velocity of the pendulum is zero the pendulum can be moved in an arbitrary direction only if $|\theta| \leq \theta_0$ or $|\theta - \pi| \leq \theta_0$, where $\theta_0 = \arctan n$. This set depends critically on the parameter n. It set covers an angle of $4\theta_0$, $n = 10$ gives $4\theta_0 = 337°$ and $n = 0.1$ gives $4\theta_0 = 23°$.

3. ENERGY CONTROL

Many tasks can be accomplished by controlling the energy of the pendulum instead of controlling its position and velocity directly, see Wiklund et al. (1993). For example one way to swing the pendulum to the upright position is to give it an energy that corresponds to the upright position. The pendulum will then swing and it can then be caught in the upright position with a stabilizing strategy.

The energy E of the uncontrolled pendulum is given by Equation (2). To perform energy control it is necessary to understand how the energy is influenced by the acceleration of the pivot. Computing the derivative of E with respect to time we find

$$\frac{dE}{dt} = J_p \dot{\theta} \ddot{\theta} - mgl\dot{\theta} \sin \theta = -mul\dot{\theta} \cos \theta \qquad (3)$$

It follows from Equation (3) that it is easy to control the energy. The system is simply an integrator with varying gain. Controllability is lost when the coefficient of u in the right hand side of (3) vanishes. This occurs for $\dot{\theta} = 0$ or $\theta = \pm\pi/2$, i.e., when the pendulum is horizontal or when it reverses its velocity. Control action is most effective when the angle θ is 0 or π and the velocity is large. To increase energy the acceleration of the pivot u should be positive when the quantity $\dot{\theta} \cos \theta$ is negative. To change the energy as fast as possible the magnitude of the control signal should be as large as possible.

Let the desired energy be E_0. The following control law is a simple strategy for achieving the desired energy:

$$u = \mathrm{sat}_{ng} \left(k(E - E_0) \right) \mathrm{sign}(\dot{\theta} \cos \theta) \qquad (4)$$

where k is a design parameter. In this expression the function sat_{ng} denotes a function which saturates at ng. This strategy is essentially a bang-bang strategy for large errors and a proportional control for small errors. It will drive the energy towards the desired value. The parameter n is crucial because it gives the maximum control signal and thus the maximum rate of energy increase. Compare with Equation (3). For large values of k the strategy (4) is arbitrarily close to the strategy that gives the maximum increase of energy.

4. SWING UP STRATEGIES

We will now discuss strategies for bringing the pendulum to the upright position. It is assumed that the pendulum starts at rest in the stable downward position. The energy of the pendulum is defined to be $-2mgl$ initially. In the steady state upright position the energy is zero. It is very convenient to use arguments based on control of the energy of the pendulum. One way to swing up the pendulum is to control it so that its energy increases from $-2mgl$ to zero. The pendulum then swings towards the upright position where it can be captured by a stabilizing strategy. It can be shown that strategies of this type will converge to the desired solution, see Fradkov (1991) and Fradkov et al. (1995). A very simple strategy is obtained in the following way. Assume that the pendulum is at rest in the downward position. Accelerate with maximum acceleration in an arbitrary direction and reverse the acceleration when the velocity becomes zero. To see what happens with such a strategy we introduce a coordinate system fixed to the pivot point. In this coordinate system the center of the mass of the pendulum moves in a circle as indicated in Figure 1. If the acceleration is towards the right there is a field in the direction OB with an apparent gravity of magnitude $n\sqrt{1 + n^2}$ in the direction OB. The pendulum then swings symmetrically around OB. Consider, for example, the situation when the pendulum has zero velocity at the point A. The pendulum then swings to the position C where the angle is $\varphi + 2\theta_0$, indicated by C in Figure 1 where the velocity becomes zero. The pendulum thus increases its swing by an amount corresponding to the angle $2\theta_0$ for each reversal of the velocity. Notice that this simple strategy is optimal only if the pendulum does not reach the horizon, because it follows from Equation (4) that the acceler-
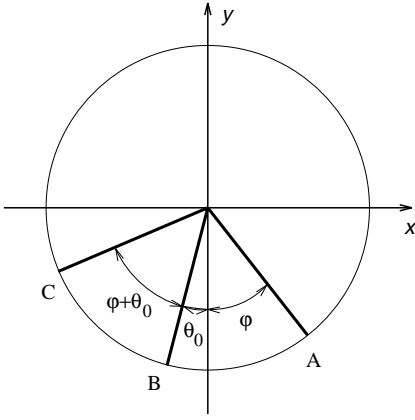
Fig. 1 Geometric illustration of a simple swing-up strategy.

Fig. 2 Simulation of a single-swing double-switch strategy. The parameters are $n = 2.1$, $\omega_0 = 1$ and $k = 100$.
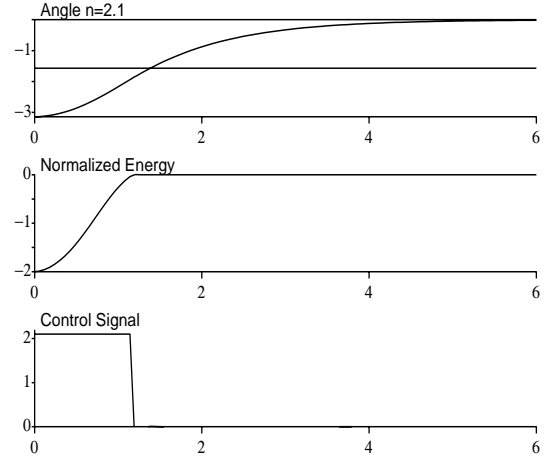
ation should be reversed when the pendulum is horizontal. Also notice that it must be required that it is possible to maintain maximum acceleration during the full swing.

5. SINGLE-SWING BEHAVIOR

The behavior of the swing up depends critically on the maximum acceleration of the pivot. If the parameter n is sufficiently large the required energy is obtained very quickly, and the pendulum can be brought up in one swing, but many swings are required if n is small.

Double-Switch Control

If the available acceleration is sufficiently large, the pendulum can be swung up simply by using the maximum acceleration until the desired energy is obtained and then setting acceleration to zero. With this strategy the control signal switches from zero to its largest value and then back to zero again. This motivates the name of the strategy. To find this strategy we will consider a coordinate system fixed to the pivot of the pendulum and regard the force due to the acceleration of the pivot as an external force. In this coordinate system the center of mass of the pendulum moves along a circular path with radius l. It follows from Equation (4) that the desired energy must be reached before the pendulum reaches the horizontal position. If the acceleration of the pivot is kept at its maximum value ng until it reaches horizontal, the energy supplied to the pendulum by the acceleration of the pivot is $nmgl$. To swing up the pendulum its energy must be increased by $2mgl$ and we thus find that the maximum acceleration must be at least $2g$. When the acceleration is $2g$ the pendulum is accelerated until it is horizontal and

the control signal is then zero. If the acceleration is larger than $2g$ the acceleration will be switched off when the pendulum has swung the angle ϕ. The energy supplied to the pendulum is then $nmgl \sin \phi$. Equating this with $2mgl$ gives $\sin \phi = 2/n$. The single-switch double-switch strategy is illustrated in Figure 2 which shows the angle, the angular velocity, and the control signal for the case $n = 2.1$. Notice that maximum control action is used until the desired energy is reached. This happens shortly before the pendulum is horizontal.

Large Accelerations

If the available acceleration is much larger than $2g$, it is possible to find approximate expressions for the switching times. We will consider the case when the pendulum starts in the downward position. If the acceleration is so large that the desired energy can be obtained for small angles the equation of motion (1) can be approximated by
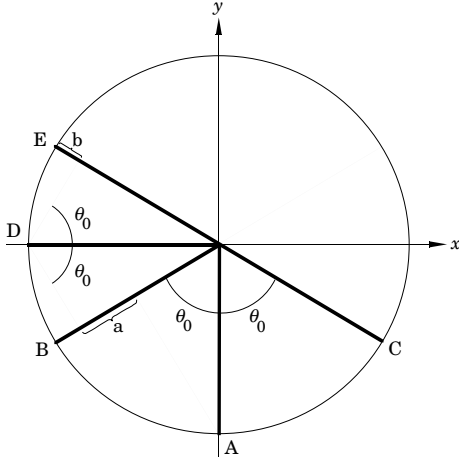
$$J_p \frac{d^2\theta}{dt^2} = mnlg \qquad (5)$$

This equation has the solution $\theta = (\omega_0 t)^2/2$. The energy increase of the pendulum is thus

$$E = mngl \sin \theta \approx mngl\theta = mgl \, \frac{(\omega_0 t)^2}{2}$$

Requiring this to be equal to the swing-up energy $2mgl$ we find that the time for full acceleration is

$$t \approx \frac{2}{n\omega_0}$$

where ω_0 is the frequency of small oscillations of the pendulum. The pendulum changes the angle $\theta \approx 2/n$,

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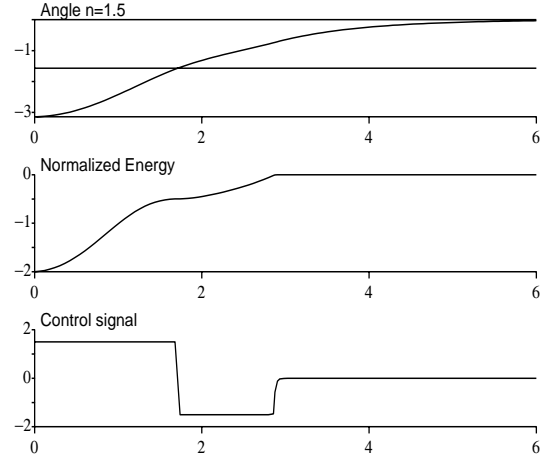**Fig. 3** Diagram used to explain the single-swing, triple-switch behavior.

Fig. 4 Simulation of the single-swing triple-switch control. The parameters are $n = 1.5$, $\omega_0 = 1$ and $k = 100$.

compare the exact value $\arctan 2/n$. With $n = 10$ we find, e.g., that the angle changes with about $11.5°$ during the acceleration required for swing-up.

Triple-Switch Control

In the single-swing strategy discussed above the pendulum swings to one side and continues to swing until it reaches the upright position. It is possible to find strategies where the pendulum behaves in the same way even when the maximum acceleration is smaller than $2g$. The control signal will, however, switch several times. The acceleration must be larger than g in order for the pendulum to reach the horizontal in one swing. To find out how much larger it has to be we will consider the situation illustrated in Figure 3. The pendulum starts at rest at position A, maximum acceleration ng is first applied in the positive direction. An observer fixed to the pivot sees a gravitational field in the direction OB with the strength

$$w = \sqrt{u^2 + g^2}$$

When the pendulum moves from A to D it loses the potential energy $mwa = mw(\sin \theta_0 - \cos \theta_0)$, which is converted to kinetic energy. It follows from Equation (3) that the control acceleration of the pivot should be reversed when the pendulum is horizontal in order to maximize the energy given to the pendulum. When the acceleration is reversed an observer fixed to the pivot sees a gravitational field in the direction OC. The kinetic energy of the pendulum must then be sufficiently large to bring the pendulum to the position E. The difference in potential energy is mwb. We thus obtain the condition $a > b$. It follows from Figure 3 that $a = \sin \theta_0 - \cos \theta_0$ and $b = 1 - \sin \theta_0$

which gives

$$\sin \theta_0 - \cos \theta_0 \geq 1 - \sin \theta_0 \qquad (6)$$

Introducing $n = \tan \theta_0$ and using equality in the above equation we get

$$2n = \sqrt{1 + n^2} + 1$$

This equation has the solution $n = 4/3$. The acceleration must thus be at least $4g/3$ to have a single-swing behavior. When the acceleration is exactly equal to $4g/3$ the pivot will first accelerate to the right until the pendulum reaches the horizontal. The acceleration is then reversed until the pendulum reaches the desired energy when the acceleration is set to zero. When n is greater than $4/3$ the acceleration of the pivot can be set to zero before the pendulum reaches the point E. Let the pendulum form the angle ϕ with the horizontal when the acceleration of the pivot is set to zero. The energy supplied to the pendulum is then $mgl(2 - \sin \phi)$, because the pivot moves the distance $l + L(1 - \sin \phi)$ with acceleration ng. Equating this with $2mgl$ gives

$$\sin \phi = \frac{2 - n}{n}$$

For $n = 4/3$ we get $\phi = 30°$.

The single-swing triple-switch control is illustrated in Figure 4. The maximum control signal is applied initially. Energy increases but it has not reached the desired level when the pendulum is horizontal. To continue to supply energy to the pendulum the control signal is then reversed until the desired energy is obtained. The control signal is then set to zero. Since $n = 2.1$ is close to the limit $n = 2$ for this type of behavior, the pendulum is close to the desired upright position when the control signal is set to zero.

6. MULTI-SWING BEHAVIOR

If the maximum acceleration is smaller than $4g/3$ it is necessary to swing the pendulum several times before it reaches the upright position. Let us first consider the conditions for bringing the pendulum up in two swings illustrated in Figure 5, which shows a coordinate system fixed to the pivot. An observer in this coordinate system sees a field with strength $w = \sqrt{g(1+n^2)}$. The field has direction OB if the acceleration of the pivot is positive, and the direction OC when it is negative. Assume that the pendulum starts at rest at A and that maximum acceleration is first given in the positive direction. The pendulum then swings from A to C in the figure. At that point the pendulum turns and the acceleration of the pivot is reversed. When the pendulum reaches the horizontal position at E the acceleration of the pivot is again reversed. The pendulum will reach the upright position if the velocity at E is sufficiently large. The pendulum starts at rest at D. When it has moved to E it has lost the potential energy mwa, which has been transferred to kinetic energy. This kinetic energy must be sufficiently large to move the pendulum to F. This energy is mwb and we get the condition $b \geq a$. It follows from Figure 5 that $a = \sin\theta_0 - \cos 3\theta_0$ and $b = 1 - \sin\theta_0$. Hence

$$\sin\theta_0 - \cos 3\theta_0 \geq 1 - \sin\theta_0$$

or
$$2\sin\theta_0 = 1 + \cos 3\theta_0 \tag{7}$$

This equation has the solution $\theta_0 = 30°$. The maximum acceleration required is thus $0.58g$. The control signal switches four times in this strategy because it is necessary to reverse control at D in order to reduce the energy of the pendulum.

The General Case

It is easy to extend the argument to cases where more swings are required. For example in a strategy with three swings the pendulum first swings $2\theta_0$ in one direction. Next time it swings $4\theta_0$ in the other direction, and the condition to reach the upright position becomes
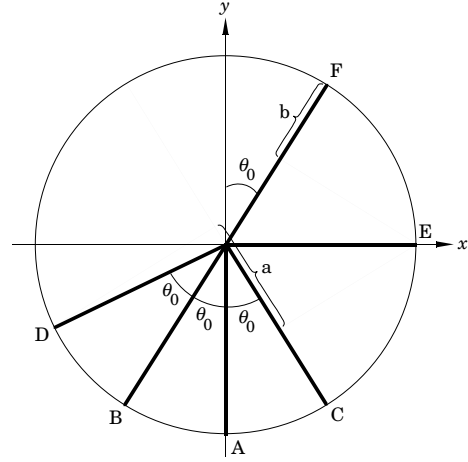
$$2\sin\theta_0 \geq 1 + \cos 5\theta_0$$
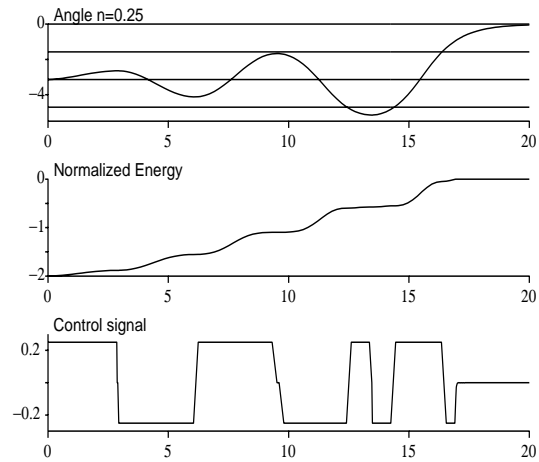
A case with k swings gives the equation

$$2\sin\theta_0 \geq 1 + \cos(2k-1)\theta_0 \tag{8}$$

Solving this equation numerically we obtain the the following values:

n	1.333	0.577	0.388	0.296	0.241	0.128
k	1	2	3	4	5	10

Fig. 5 Figure used to derive the conditions for the double-swing behavior.

Fig. 6 Simulation of energy control for the case $n = 0.25$, $\omega_0 = 1$ and $k = 100$. Five swings are required in this case.

For small values of n the relation between n and k is approximately given by

$$k \approx \frac{\pi}{2k-1}$$

The number of switches required increases with decreasing n. With $n = 0.25$ we find that five switches are required to swing up the pendulum. This is illustrated in Figure 6.

7. MINIMUM TIME STRATEGIES

It follows from Pontryagins maximum principle that the minimum time strategies for swinging up the pendulum are of bang-bang type. It can be shown that the strategies have a nice interpretation as energy control. They will inject energy into the pendulum at maximum rate and then remove energy at maximum rate in such a way that the energy corresponds to
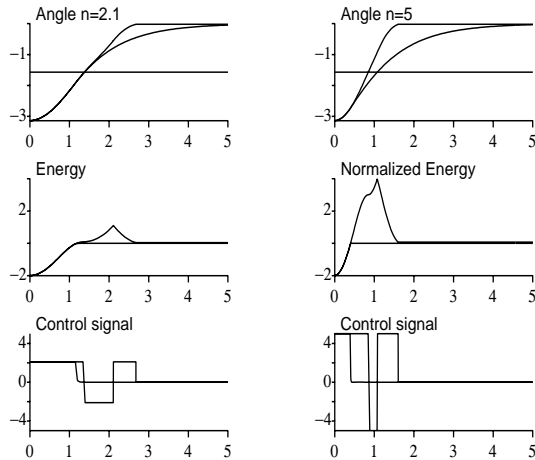
Fig. 7 A comparison of energy control with minimum time control for $n = 2.1$ (left) and $n = 5$ (right).

the equilibrium energy when the upright position is reached. For small values of n the minimum time strategies give control signals that are identical with the strategies based on energy control initially. The final part of the control signals are, however, different because the strategies we have described will set the control signal to zero when the desired energy has been obtained. The strategies we have given can thus be described as strategies where there is no overshoot in the energy.

Consider for example the case $n > 2$, where a one-swing strategy can be used. To swing up the pendulum its energy must be increased with $2mgl$. This can be achieved by a single-swing double-switch strategy illustrated in Figure 2. The maximum acceleration is used until the pendulum has moved the angle $\arctan 2/n$. The energy can be increased further by continuing the acceleration, until the pendulum has reached the horizontal position. It follows from Equation (3) that the acceleration should then be reversed. By reversing the acceleration at a proper position the energy can then be reduced so that it reaches the desired value when the pendulum is horizontal. Working out the details we find that the maximum energy is obtained for the angle

$$\arcsin 1 - 1/n$$

The energy is then reduced at the maximum rate. Figure 7 compares the minimum time strategies and the energy control strategies. The figure also shows that the difference in the time to reach the upright position increases with increasing n. It also shows that the minimum time strategy has an overshoot in the energy. The overshoot increases with increasing n. For the case $n = 5$ the overshoot is more than

200%. This explains why the minimum time strategies are quite sensitive. The excess energy is dissipated rapidly when the pendulum approaches the upright position. The energy control gives a much gentler approach towards the upright position. Several different strategies are often combined to swing up the pendulum. A catching strategy is used when the pendulum is close to the upright position. A good practical approach is to use an energy control strategy with an energy excess of 10–20% and catch the pendulum when it is close to the upright position. Such a strategy is simple and quite robust to modeling errors.

8. CONCLUSION

It has been shown that it is very convenient to swing up a pendulum by controlling its energy. The control strategy, which is given by Equation (4), is very simple. The behavior obtained with this strategy depends critically on one parameter, the available acceleration. If the acceleration is sufficiently large, $u > 2g$ the pendulum can be brought to the upright position with one swing. The control signal uses its maximum value until the desired energy is obtained and is then set to zero. For $u > 4g/3$ the pendulum can still be brought up with one swing, but the control signal now makes three switches. For lower accelerations the pendulum has to swing several times.

9. REFERENCES

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